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**OPERATOR ASPECTS, SYSTEMS OF Q -DIFFERENCE EQUATIONS
AND Q -INTEGRALS FOR DIFFERENT TYPES OF MULTIPLE
 Q -FUNCTIONS IN THE SPIRIT OF HORN, DEBIARD AND GAVEAU**

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Abstract

The purpose of this article is to continue the studies of a new complete multiple q -hypergeometric symbolic calculus, which leads to q -Euler integrals and a very similar canonical system of q -difference equations for q -Appell, q -Horn and confluent multiple q -functions. We present the systems of differential equations for Horn functions, and, accordingly, simplify the canonical q -difference equations which are very similar for $q = 1$. In order to define the q -Euler integrals, we have to use different operator definitions of some q -Horn functions, whose q -integral expressions are proved. The previous concepts normal and abnormal q -Appell functions thus appear again for our q -Horn functions.

The second part of the paper is devoted to confluent q -Appell functions and confluent q -Horn functions.

2010 Mathematics Subject Classification: Primary 33D70; Secondary 33C65

Keywords: symbolic calculus, canonical system of q -difference equations, q -Euler integral, confluent q -hypergeometric function, q -Appell function, q -Horn function

ASPECTS OPÉRATEURS, SYSTÈMES D'ÉQUATIONS AUX
 Q -DIFFÉRENCES ET Q -INTÉGRALES POUR DIFFÉRENTS
TYPES DE Q -FONCTIONS MULTIPLES DANS L'ESPRIT DE
HORN, DEBIARD ET GAVEAU

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SOMMAIRE. Le but de cet article est de poursuivre les études d'un nouveau calcul complet et symbolique pour la série q -hypergéométrique multiple, menant aux q -intégrales d'Euler et à un système canonique similaire aux équations de q -différence pour les q -fonctions de q -Appell, q -Horn et des q -fonction multiples confluentes. Nous allons présenter les systèmes des équations de différence pour les fonctions de Horn, et, ensuite, simplifier les équations canoniques de q -différence qui sont très pareilles à $q = 1$. Afin que nous puissions définir les q -intégrales d'Euler, il faut que nous appliquions les diverses définitions opérationnelles de certaines fonctions de q -Horn, dont les q -intégrales sont prouvées. Les notations précédente, les fonctions de q -Appel normal et anormal, apparaîtront encore une fois pour nos fonctions de q -Horn.

La deuxième partie de ce texte est dédiée aux fonctions de q -Appell confluentes et aux fonctions confluentes de q -Horn.

Classification mathématique par matières 2010: Primaire 33D70; Secondaire 33C65

Mots clés: calcul symbolique, système canonique aux équations de q -différence, q -intégrale d'Euler, q -fonction multiples confluyente, fonction de q -Appell, fonction de q -Horn

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1. INTRODUCTION, DÉFINITIONS ET FORMULES DE LIMITE

Voici le troisième texte de trois articles pareils sur les fonctions q -hypergéométriques multiples et/ou confluentes. Le premier article était [9], où on a étudié les formules de Jackson, Feldheim et Burchall-Chaundy pour les fonctions q -hypergéométriques confluentes. Le deuxième article était [10], où cet calcul d'opérateur a été présenté et appliqué aux quatre fonctions de q -Appell. Contrairement aux fonctions hypergéométriques, ces fonctions confluentes se forme simplement en laissant l'un des paramètres tendre vers l'infini. Pour la commodité de lectrice et lecteur, nous allons indiquer les systèmes d'équations de différence pour les fonctions de Horn, puisque la notation et les formules de l'article révolutionnaire [3, p. 816 et suiv.], qui fait époque dans la science mathématique, sont vagues et insuffisantes. Nous voulons faire observer que les formes d'équation aux dérivées partielles et les formes opérationnelles que nous avons présentées ici sont totalement différentes.

Les fonctions de Horn sont déjà bien connues, et donc nous définirons dix q -analogues de séries de deux variables de Horn, dont les régions de convergence très pareilles et étendues seront publiées ailleurs. D'abord nous allons présenter les définitions et notations nombreuses.

1.1. Notation du calcul q -logarithmique. D'abord quelques commentaires généraux sur la notation à utiliser. Dans cet article entier, le symbole $\equiv$ dénotera les définitions, sauf quand nous nous occupons des congruences. Nous allons écrire $\doteq$ pour indiquer l'ordre symbolique (ombrage). La notation $\cong$ signifie égalité formelle. Nous allons indiquer le côté droit (d'une équation) par LCD et le côté gauche par LCG.

Nous allons tourner vers le sujet de ce texte, le calcul q , inventé par Euler, Gauss et Cauchy.

Dans le calcul q nous allons chercher les q -analogues des objets mathématique, qui ayant des limites de l'objet original quand $q \rightarrow 1$.

Définition 1. La fonction puissance est définie par $q^a \equiv e^{a \log(q)}$. Soit $\delta > 0$ un petit nombre arbitraire. Nous allons toujours employer la branche du logarithme: $-\pi + \delta < \text{Im}(\log q) \leq \pi + \delta$.

Le q -analogue d'un nombre complexe a est défini par:

$$\{a\}_q \equiv \frac{1 - q^a}{1 - q}, \quad q \in \mathbb{C} \setminus \{1\}. \quad (1)$$

Les q -factorielles sont données par

$$\langle a; q \rangle_n \equiv \begin{cases} 1, & \text{si } n = 0; \\ \prod_{m=0}^{n-1} (1 - q^{a+m}), & \text{si } n = 1, 2, \dots, \end{cases} \quad (2)$$

$$(a; q)_n \equiv \prod_{m=0}^{n-1} (1 - aq^m). \quad (3)$$

Ensuite,

$$\langle \infty; q \rangle_n \equiv 1, \quad 0 < |q| < 1. \quad (4)$$

$$\langle a; q \rangle_\infty \equiv \prod_{m=0}^{\infty} (1 - q^{a+m}), \quad 0 < |q| < 1. \quad (5)$$

Si $|q| > 1$, ou

$0 < |q| < 1$ et $|z| < 1$,

la petite fonction q -exponentielle $e_q(z)$ est donnée par

$$e_q(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\langle 1; q \rangle_k} z^k. \quad (6)$$

La petite fonction q -exponentielle est utilisée dans les formules (130), (139), (142)- (144), (146), (258) et (259).

Définition 2. Dans la suite, $\frac{\mathbb{C}}{\mathbb{Z}}$ denotera l'espace des nombres complexes mod $\frac{2\pi i}{\log q}$. Cela est isomorphe au cylindre $\mathbb{R} \times e^{2\pi i \theta}$, $\theta \in \mathbb{R}$.

L'opérateur

$$\sim: \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

est défini par

$$a \mapsto a + \frac{\pi i}{\log q}. \quad (7)$$

De (7) il suit que

$$\widetilde{\langle a; q \rangle_n} = \prod_{m=0}^{n-1} (1 + q^{a+m}), \quad (8)$$

où le tilde dénote cette fois une involution qui change un signe moins en un signe plus dans tous les n facteurs de $\langle a; q \rangle_n$.

Définition 3. En généralisant la série de Heine, nous allons définir une série q -hypergéométrique par:

$$\begin{aligned} {}_p\phi_r(\hat{a}_1, \dots, \hat{a}_p; \hat{b}_1, \dots, \hat{b}_r | q, z) &\equiv {}_p\phi_r \left[\begin{matrix} \hat{a}_1, \dots, \hat{a}_p \\ \hat{b}_1, \dots, \hat{b}_r \end{matrix} \middle| q, z \right] = \\ &= \sum_{n=0}^{\infty} \frac{\langle \hat{a}_1, \dots, \hat{a}_p; q \rangle_n}{\langle 1, \hat{b}_1, \dots, \hat{b}_r; q \rangle_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+r-p} z^n, \quad \hat{a} \equiv a \vee \tilde{a}. \end{aligned} \quad (9)$$

La définition (9) peut être généralisée à

$$\begin{aligned} {}_{p+p'}\phi_{r+r'} \left[\begin{matrix} \hat{a}_1, \dots, \hat{a}_p \\ \hat{b}_1, \dots, \hat{b}_r \end{matrix} \middle| q; z \middle| \begin{matrix} \prod_i f_i(k) \\ \prod_j g_j(k) \end{matrix} \right] &\equiv \\ \sum_{k=0}^{\infty} \frac{\langle \hat{a}_1; q \rangle_k \dots \langle \hat{a}_p; q \rangle_k}{\langle 1, \hat{b}_1; q \rangle_k \dots \langle \hat{b}_r; q \rangle_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+r+r'-p-p'} z^k \frac{\prod_i f_i(k)}{\prod_j g_j(k)}, \end{aligned} \quad (10)$$

où

$$\hat{a} \equiv a \vee \tilde{a}. \quad (11)$$

Nous supposons que le dénominateur ne contient aucun facteur nul. Nous supposons que les $f_i(k)$ et $g_j(k)$ contiennent des p' et r' facteurs de la forme de $\langle a(k); q \rangle_k$ ou $\langle s(k); q \rangle_k$ respectivement.

La notation suivante sera convenable.

$$\text{QE}(x) \equiv q^x. \quad (12)$$

Définition 4. [7, 1.45] La fonction Γ_q est définie par

$$\Gamma_q(z) \equiv \begin{cases} \frac{\langle 1; q \rangle_{\infty}}{\langle z; q \rangle_{\infty}} (1-q)^{1-z} & \text{si } 0 < |q| < 1 \\ \frac{\langle 1; q^{-1} \rangle_{\infty}}{\langle x; q^{-1} \rangle_{\infty}} (q-1)^{1-x} q^{\binom{x}{2}}, & \text{si } |q| > 1. \end{cases} \quad (13)$$

Définition 5. [7, 1.49] Soit S_r les pôles additionnels de Γ_q , vertical si q est réel et oblique si q est complexe. Alors la fonction Γ_q généralisée, une fonction $(\mathbb{C} \setminus (\{\mathbb{Z} \leq 0\} \cup S_r))^{p+r} \times \mathbb{C} \mapsto \mathbb{C}$, est défini comme suit:

$$\Gamma_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_r \end{matrix} \right] \equiv \frac{\Gamma_q(a_1) \dots \Gamma_q(a_p)}{\Gamma_q(b_1) \dots \Gamma_q(b_r)}. \quad (14)$$

Définition 6. La dérivée q est définie par

$$(D_q \varphi)(x) \equiv \begin{cases} \frac{\varphi(x) - \varphi(qx)}{(1-q)x}, & \text{si } q \in \mathbb{C} \setminus \{1\}, x \neq 0; \\ \frac{d\varphi}{dx}(x), & \text{si } q = 1; \\ \frac{d\varphi}{dx}(0), & \text{si } x = 0. \end{cases} \quad (15)$$

Si nous voulons indiquer la variable que l'opérateur de différence q s'applique, on écrit $(D_{q,x} \varphi)(x, y)$ pour l'opérateur.

Définition 7. L'analogie q de l'opérateur d'Euler $x_j D_{x_j}$ est défini par

$$\theta_{q,j} \equiv x_j D_{q,x_j}. \quad (16)$$

Définition 8. La q -intégrale [7, 6.52] est l'inverse du q -dérivée.

$$\int_a^b f(t, q) d_q(t) \equiv \int_0^b f(t, q) d_q(t) - \int_0^a f(t, q) d_q(t), \quad a, b \in \mathbb{R}, \quad (17)$$

où

$$\int_0^a f(t, q) d_q(t) \equiv a(1-q) \sum_{n=0}^{\infty} f(aq^n, q) q^n, \quad 0 < |q| < 1, \quad a \in \mathbb{R}. \quad (18)$$

Théorème 1.1. Une intégrale q -Euler pour ${}_2\phi_1$ [7, 7.50]

$${}_2\phi_1(a, b; c|q; z) \cong \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 \frac{t^{b-1} (qt; q)_{c-b-1}}{(zt; q)_a} d_q(t). \quad (19)$$

1.2. Les fonctions q -hypergéométriques confluentes de Appell. Puisque le nombre de factorielles q échangés aux dénominateurs est plus grand que cela aux numérateurs pour les fonctions q -confluentes, par le critère du quotient, les régions de convergence sont considérablement augmentées. Ces régions de convergence dans le cas hypergéométrique sont seulement données par Srivastava et Karlsson dans [13].

Définition 9. Les fonctions q -hypergéométriques doubles confluentes [9] sont

$$\Upsilon_1(a; b; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1+m_2} \langle b; q \rangle_{m_1}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2}, \quad (20)$$

$$|x_1| < 1, \quad |(1-q)x_2| < \infty,$$

$$\Psi_1(a; b; c, c'|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1+m_2} \langle b; q \rangle_{m_1}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1} \langle c'; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}, \quad (21)$$

$$|x_1| < 1, \quad |(1-q)x_2| < \infty,$$

$$\Xi_1(a, a'; b; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1} \langle a'; q \rangle_{m_2} \langle b; q \rangle_{m_1}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2}, \quad (22)$$

$$|x_1| < 1, \quad |(1-q)x_2| < \infty,$$

$${}_2\phi_1(a, b; c|q; x_1) {}_2\phi_1(a', \infty; c'|q; x_2)$$

$$\equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a, b; q \rangle_{m_1} \langle a'; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1} \langle c'; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}, \quad |x_1| < 1, \quad |(1-q)x_2| < \infty, \quad (23)$$

$$\Upsilon_2(a, a'; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1} \langle a'; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2}, \quad (24)$$

$$|(1-q)x_1| < \infty, \quad |(1-q)x_2| < \infty,$$

$$\Psi_2(a; c, c'|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1+m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1} \langle c'; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}, \quad (25)$$

$$|(1-q)x_1| < \infty, \quad |(1-q)x_2| < \infty,$$

$$\Psi_3(a; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1+m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1}} x_1^{m_1} x_2^{m_2}, \quad (26)$$

$$|(1-q)x_1| < \infty, \quad |(1-q)x_2| < \infty,$$

$$\begin{aligned}
& {}_2\phi_1(a, \infty; c|q; x_1) {}_2\phi_1(a', \infty; c'|q; x_2) \\
& \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1} \langle a'; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1} \langle c'; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}, \quad (27) \\
& |(1-q)x_1| < \infty, \quad |(1-q)x_2| < \infty,
\end{aligned}$$

$$\begin{aligned}
& \Xi_2(a; b; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1} \langle b; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2}, \quad (28) \\
& |x_1| < 1, \quad |(1-q)^2 x_2| < \infty,
\end{aligned}$$

$$\begin{aligned}
& {}_2\phi_1(a, b; c|q; x_1) {}_2\phi_1(2\infty; c'|q; x_2) \\
& \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a, b; q \rangle_{m_1}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1} \langle c'; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}, \quad |x_1| < 1, \quad |(1-q)^2 x_2| < \infty, \quad (29)
\end{aligned}$$

$$\begin{aligned}
& \Upsilon_3(a; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2}, \quad (30) \\
& |(1-q)x_1| < \infty, \quad |(1-q)^2 x_2| < \infty.
\end{aligned}$$

1.3. Les premières fonctions de q -Horn. Probablement, les fonctions de q -Horn n'ont pas été définies auparavant; leurs régions de convergence sont plus grandes que les fonctions correspondantes de Horn.

Définition 10.

$$G_1(a; b; b'|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1+m_2} \langle b; q \rangle_{m_2-m_1} \langle b'; q \rangle_{m_1-m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}. \quad (31)$$

$$\begin{aligned}
& G_2(a; a'; b, b'|q; x_1, x_2) \\
& \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1} \langle a'; q \rangle_{m_2} \langle b; q \rangle_{m_2-m_1} \langle b'; q \rangle_{m_1-m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}. \quad (32)
\end{aligned}$$

$$G_3(a; a'|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{2m_2-m_1} \langle a'; q \rangle_{2m_1-m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}. \quad (33)$$

$$H_1(a; b; c; d|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1-m_2} \langle b; q \rangle_{m_1+m_2} \langle c; q \rangle_{m_2}}{\langle 1, d; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}. \quad (34)$$

$$H_2(a; b; c; d; e|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1-m_2} \langle b; q \rangle_{m_1} \langle c, d; q \rangle_{m_2}}{\langle 1, e; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}. \quad (35)$$

$$H_3(a; b; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{2m_1+m_2} \langle b; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2} \langle c; q \rangle_{m_1+m_2}} x_1^{m_1} x_2^{m_2}. \quad (36)$$

$$H_4(a; b; c; d|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{2m_1+m_2} \langle b; q \rangle_{m_2}}{\langle 1, c; q \rangle_{m_1} \langle 1, d; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}. \quad (37)$$

$$H_5(a; b; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{2m_1+m_2} \langle b; q \rangle_{m_2-m_1}}{\langle 1; q \rangle_{m_1} \langle 1, c; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}. \quad (38)$$

$$H_6(a; b; c|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{2m_1-m_2} \langle b; q \rangle_{m_2-m_1} \langle c; q \rangle_{m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}. \quad (39)$$

$$H_7(a; b; c; d|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{2m_1-m_2} \langle b, c; q \rangle_{m_2}}{\langle 1, d; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} x_1^{m_1} x_2^{m_2}. \quad (40)$$

1.4. Les fonctions de q -Horn confluentes (normales). Dans [7, p. 366] nous avons souligné que pour les quatre (soi-disant) fonctions de q -Appell anormales, $x_2^{m_2} q^{\binom{m_2}{2}}$ remplace $x_2^{m_2}$ dans le terme général.

On verra ça que les fonctions normales (dans un sens plus large) correspondent au q -addition NWA, et les fonctions anormales correspondentes au q -addition JHC.

Définition 11. Les fonctions (normales) confluentes de q -Horn, q -analogues de [12], [2], [4], sont définis par

$$\Gamma_1(a; b, b'|q; x_1, x_2) \equiv \lim a' \rightarrow \infty G_2(a; a'; b, b'|q; x_1, x_2). \quad (41)$$

$$\Gamma_2(b, b'|q; x_1, x_2) \equiv \lim a, a' \rightarrow \infty G_2(a; a'; b, b'|q; x_1, x_2). \quad (42)$$

$$\mathcal{H}_1(a; b; d|q; x_1, x_2) \equiv \lim c \rightarrow \infty H_1(a; b; c; d|q; x_1, x_2). \quad (43)$$

$$\mathcal{H}_2(a; b; d; e|q; x_1, x_2) \equiv \lim c \rightarrow \infty H_2(a; b; c; d; e|q; x_1, x_2). \quad (44)$$

$$\mathcal{H}_3(a; b; e|q; x_1, x_2) \equiv \lim c, d \rightarrow \infty H_2(a; b; c; d; e|q; x_1, x_2). \quad (45)$$

$$\mathcal{H}_4(a; d; e|q; x_1, x_2) \equiv \lim b, c \rightarrow \infty H_2(a; b; c; d; e|q; x_1, x_2). \quad (46)$$

$$\mathcal{H}_5(a; d|q; x_1, x_2) \equiv \lim b, c \rightarrow \infty H_1(a; b; c; d|q; x_1, x_2). \quad (47)$$

$$\mathcal{H}_6(a; c|q; x_1, x_2) \equiv \lim b \rightarrow \infty H_3(a; b; c|q; x_1, x_2). \quad (48)$$

$$\mathcal{H}_7(a; c; d|q; x_1, x_2) \equiv \lim b \rightarrow \infty H_4(a; b; c; d|q; x_1, x_2). \quad (49)$$

$$\mathcal{H}_8(a; b|q; x_1, x_2) \equiv \lim c \rightarrow \infty H_6(a; b; c|q; x_1, x_2). \quad (50)$$

$$\mathcal{H}_9(a; c; d|q; x_1, x_2) \equiv \lim b \rightarrow \infty H_7(a; b; c; d|q; x_1, x_2). \quad (51)$$

$$\mathcal{H}_{10}(a; d|q; x_1, x_2) \equiv \lim b, c \rightarrow \infty H_7(a; b; c; d|q; x_1, x_2). \quad (52)$$

$$\mathcal{H}_{11}(a; c; d; e|q; x_1, x_2) \equiv \lim b \rightarrow \infty H_2(a; b; c; d; e|q; x_1, x_2). \quad (53)$$

1.5. Les fonctions de q -Horn confluentes (anormale). Pour définir les fonctions (anormale) confluentes de q -Horn, nous avons besoin d'une notation pour simplifier les formules.

Définition 12. Nous définissons la forme quadratique suivante par les nombres positifs m et n .

$$\mathcal{Q}(m, n, a) \equiv -\binom{n}{2} + n(m + a - 1). \quad (54)$$

Pour les définitions suivantes, les formules d'opérateur dans [3] demandent des facteurs q de plus. Notez que l'indice premier ressemble à la fonction correspondante de Horn, et le deuxième indice dénote les variantes avec les q -facteurs différents.

Définition 13. Les fonctions de q -Horn (anormale) sont définis par

$$\begin{aligned} G_{21}(a; a'; b, b'|q; x_1, x_2) &\equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1} \langle a'; q \rangle_{m_2} \langle b; q \rangle_{m_2-m_1} \langle b'; q \rangle_{m_1-m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} \\ &\text{QE} (\mathcal{Q}(0, m_2, b') + \mathcal{Q}(m_2, m_1, b)) x_1^{m_1} x_2^{m_2}. \end{aligned} \quad (55)$$

$$\begin{aligned} H_{11}(a; b; c; d|q; x_1, x_2) &\equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1-m_2} \langle b; q \rangle_{m_1+m_2} \langle c; q \rangle_{m_2}}{\langle 1, d; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} \\ &\text{QE} (\mathcal{Q}(0, m_2, a)) x_1^{m_1} x_2^{m_2}. \end{aligned} \quad (56)$$

$$H_{22}(a; b; c; d; e|q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1-m_2} \langle b; q \rangle_{m_1} \langle c, d; q \rangle_{m_2}}{\langle 1, e; q \rangle_{m_1} \langle 1; q \rangle_{m_2}}. \quad (57)$$

$$QE(\mathcal{Q}(0, m_2, a)) x_1^{m_1} x_2^{m_2}.$$

$$\begin{aligned} & H_{23}(a+1-e; b+1-e; c; d; 2-e|q; x_1, x_2) \\ & \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a+1-e; q \rangle_{m_1-m_2} \langle b+1-e; q \rangle_{m_1} \langle c, d; q \rangle_{m_2}}{\langle 1, 2-e; q \rangle_{m_1} \langle 1; q \rangle_{m_2}}. \quad (58) \\ & QE(\mathcal{Q}(0, m_2, a-e+1)) x_1^{m_1} x_2^{m_2}. \end{aligned}$$

Définition 14. Les fonctions $\langle$ anormale $\rangle$ confluentes de q -Horn sont définies par

$$\Gamma_{11}(a; b, b'|q; x_1, x_2) \equiv \lim_{a' \rightarrow \infty} G_{21}(a; a'; b, b'|q; x_1, x_2). \quad (59)$$

$$\Gamma_{21}(b, b'|q; x_1, x_2) \equiv \lim_{a, a' \rightarrow \infty} G_{21}(a; a'; b, b'|q; x_1, x_2). \quad (60)$$

$$\mathcal{H}_{11}(a; b; d|q; x_1, x_2) \equiv \lim_{c \rightarrow \infty} H_{11}(a; b; c; d|q; x_1, x_2). \quad (61)$$

$$\mathcal{H}_{22}(a; b; d; e|q; x_1, x_2) \equiv \lim_{c \rightarrow \infty} H_{22}(a; b; c; d; e|q; x_1, x_2). \quad (62)$$

$$\mathcal{H}_{23}(a; b; d; e|q; x_1, x_2) \equiv \lim_{c \rightarrow \infty} H_{23}(a; b; c; d; e|q; x_1, x_2). \quad (63)$$

$$\mathcal{H}_{32}(a; b; e|q; x_1, x_2) \equiv \lim_{c, d \rightarrow \infty} H_{22}(a; b; c; d; e|q; x_1, x_2). \quad (64)$$

$$\mathcal{H}_{33}(a; b; e|q; x_1, x_2) \equiv \lim_{c, d \rightarrow \infty} H_{23}(a; b; c; d; e|q; x_1, x_2). \quad (65)$$

Cet article est organisé comme suit:

Dans la section 1 nous allons présenter une introduction courte et définissons la plupart des fonctions. Dans la sous-section 1.1 nous allons présenter les fondements du calcul q -logarithmique. Dans la sous-section 1.2 nous définissons les fonctions q -Appell confluentes. Dans la sous-section 1.3 nous allons définir les fonctions premières du q -Horn. Dans la sous-section 1.4 nous allons définir les fonctions premières confluentes de q -Horn. Dans la sous-section 1.5 nous allons définir les fonctions $\langle$ anormale $\rangle$ confluentes de q -Horn.

Dans la section 2 nous introduirons le calcul général symbolique. Dans la section 3 nous allons considérer les systèmes d'équations de q -différences et des q -intégrales pour les fonctions confluentes de q -Appell. Dans la section 4 nous allons trouver d'autres solutions et leurs intégrales de q -Euler aux systèmes de équations aux q -différences pour les fonctions confluentes de

q -Appell. Dans la section 5 nous allons présenter les systèmes d'équations différentielles partielles de fonctions de Horn et leurs intégrales.

Dans la section 6 nous allons compter les équations de q -différences pour les fonctions (normale) de q -Horn.

Dans la section 7 nous allons présenter d'autres solutions pour les équations q -différence des fonctions de q -Horn.

Dans la sous-section 7.1 nous allons discuter des équations de q -différences pour $G_{2,q}$. Dans la sous-section 7.2 nous allons discuter des équations de q -différences pour $H_{2,q}$. Dans la section 8 nous allons compter les expressions opérationnelles pour les fonctions (anormale) de q -Horn avec les q -Intégrales correspondantes de q -Euler. Dans la section 9 nous allons étudier les systèmes d'équations aux q -différences pour les fonctions (normale) confluentes de q -Horn.

Dans la section 10 nous allons trouver les intégrales de q -Euler pour les fonctions (anormale) confluentes de q -Horn.

2. LE CALCUL GÉNÉRAL SYMBOLIQUE

Le but de cette section est d'introduire le calcul symbolique général pour les séries doubles.

Définition 15. Soit

$$H_q(a, b, c, x) \equiv \{\theta_{q,1} + c\}_q D_{q,x} - \{\theta_{q,1} + a\}_q \{\theta_{q,1} + b\}_q. \quad (66)$$

Ensuite nous avons

$$H_q(a, b, c, x) {}_2\phi_1 \left[\begin{matrix} a, b \\ c \end{matrix} \middle| q; x \right] = 0. \quad (67)$$

Nous sommes toujours intéressés par les solutions de l'équation

$$H_q(a, b, c, x)(f(x)) = 0. \quad (68)$$

Autour de $x = 0$ une autre solution, à part y_1 dans (67) est [7, 6.186]

$$y_2 = x^{1-c} {}_2\phi_1 \left[\begin{matrix} a - c + 1, b - c + 1 \\ 2 - c \end{matrix} \middle| q; x \right]. \quad (69)$$

Le but de la définition suivante est de conserver les puissances des variables dans l'opérateur.

Définition 16. Soient A, B, C trois opérateurs $\mathbb{R}[[x]] \rightarrow \mathbb{R}[[x]]$, qui sont linéaires dans $\theta_{q,1}$, $\theta_{q,2}$. Puis on définit

$$H_q(A, B, C, x) \equiv \{\theta_{q,1} + C\}_q D_{q,x} - \{\theta_{q,1} + A\}_q \{\theta_{q,1} + B\}_q. \quad (70)$$

Lemme 2.1. Comparer avec [3, p.777]. Soit $\mathcal{F}(a, b, c, x)$ une solution de

$$H_q(a, b, c, x)\mathcal{F} = 0. \quad (71)$$

Alors

$$\mathcal{F}(A, B, C, x) \quad (72)$$

est une solution de

$$H_q(A, B, C, x)\mathcal{F}(A, B, C, x) = 0. \quad (73)$$

Définition 17. Comparer avec [3, p.777]. Supposons $x > \bullet$, C , $\lambda \in \mathbb{R}$, $\psi(y) \times y^{-\lambda} \in \mathbb{R}[[y]]$. Puis, au sens ombral,

$$x^{C\theta_{q,2}}\psi(y) \doteq \psi(x^C y). \quad (74)$$

Suppose que

$$\phi(\vec{a}, \vec{b}, y) \equiv y^\lambda \sum_{k=0}^{\infty} \frac{\langle \vec{a}; q \rangle_k}{\langle 1, \vec{b}; q \rangle_k} y^k. \quad (75)$$

Puis, au sens ombral,

$$\phi(\vec{a} + \vec{a}'\theta_{q,2}, \vec{b} + \vec{b}'\theta_{q,2}, y) \equiv y^\lambda \sum_{k=0}^{\infty} \frac{\langle \vec{a} + \vec{a}'(\lambda + k); q \rangle_k}{\langle 1, \vec{b} + \vec{b}'(\lambda + k); q \rangle_k} y^k. \quad (76)$$

De plus, pour la fonction Γ_q :

$$\Gamma_q(\alpha + \theta_{q,2})y^\lambda \doteq \Gamma_q(\alpha + \lambda)y^\lambda. \quad (77)$$

On peut généraliser cela à de plusieurs variables.

Définition 18. Comparer avec [3, 3.9]. Soit $H_q(x, y)$ un opérateur $\mathbb{R}[[x]] \rightarrow \mathbb{R}[[x]]$:

$$H_q(x, y) \equiv \{\theta_{q,1} + \gamma\theta_{q,2} + c\}_q D_{q,x} - \{\theta_{q,1} + \alpha\theta_{q,2} + a\}_q \{\theta_{q,1} + \beta\theta_{q,2} + b\}_q. \quad (78)$$

Les paramètres dans $H_q(x, y)$ seront toujours les mêmes.

Remarque 1. La fonction (78) généralise la définition fondamentale (66) et est un cas particulier de la plus générale définition (70). La notation dans [3, 3.9] est légèrement trompeuse.

Théorème 2.2. Comparer avec [3, (3.12) p. 779]. Soit $\mathcal{F}_j(a, b, c, x)$, $j = 1, 2$ de la forme (75) sont deux solutions indépendantes de

$$H_q(a, b, c, x)\mathcal{F}_j = 0. \quad (79)$$

Par ailleurs, soit $\psi_j(y) \in \mathbb{R}[[y]]$, $j = 1, 2$ être séries q -hypergéométriques, avec des rayons de convergence convenables. Alors la solution générale de l'équation

$$H_q(x, y)f = 0 \quad (80)$$

sous la forme ombrale (76) est donnée par

$$f(x, y) = \sum_{j=1}^2 \mathcal{F}_j(\alpha\theta_{q,2} + a, \beta\theta_{q,2} + b, \gamma\theta_{q,2} + c, x)\phi_j(y). \quad (81)$$

Démonstration. Cela découle de (73). $\square$

Théorème 2.3. Comparer avec [3, (4.5) p. 780]. Soit $\psi(y) \in \mathbb{R}[[y]]$ une série q -hypergéométrique, avec un rayon de convergence convenable. Puis la série

$$F(x, y : q) \equiv {}_2\phi_1(\alpha\theta_{q,2} + a, \beta\theta_{q,2} + b, \gamma\theta_{q,2} + c|q; x)\psi(y) \quad (82)$$

est une série q -hypergéométrique double, convergente au voisinage de $(0, 0)$.

Démonstration. Similaire à [3, p. 780]. $\square$

Définition 19. Comparer avec [3, 5.1]. Présenter les deux opérateurs généraux $\mathbb{R}[[x, y]] \rightarrow \mathbb{R}[[x, y]]$:

$$\begin{aligned} H_{1,q}(x, y) &\equiv \{\gamma_1\theta_{q,2} + c_1\}_q D_{q,x} - \{\alpha_1\theta_{q,2} + a_1\}_q \{\beta_1\theta_{q,2} + b_1\}_q, \\ H_{2,q}(y, x) &\equiv \{\gamma_2\theta_{q,1} + c_2\}_q D_{q,y} - \{\alpha_2\theta_{q,1} + a_2\}_q \{\beta_2\theta_{q,1} + b_2\}_q. \end{aligned} \quad (83)$$

On souhaite étudier le système d'équations aux q -différences

$$\begin{cases} H_{1,q}(x, y)f(x, y) = 0, \\ H_{2,q}(y, x)f(x, y) = 0. \end{cases} \quad (84)$$

Le système (84) est appelé q -compatible si il a des solutions communes $f(x, y)$.

Théorème 2.4. Comparer avec [3, p. 785]. Le système (84) est q -compatible si les deux produits suivantes de q -analogues sont égaux,

$$P_1(m, n; q) = P_2(m, n; q), \quad (85)$$

où

$$\begin{aligned} P_1(m, n; q) &\equiv \{m + \alpha_1 n + a_1\}_q \{m + \beta_1 n + b_1\}_q \{m + \gamma_1(n + 1) + c_1\}_q \\ &\quad \{n + \alpha_2(m + 1) + a_2\}_q \{n + \beta_2(m + 1) + b_2\}_q \{n + \gamma_2 m + c_2\}_q, \\ P_2(m, n; q) &\equiv \{m + \alpha_1(n + 1) + a_1\}_q \{m + \beta_1(n + 1) + b_1\}_q \{m + \gamma_1 n + c_1\}_q \\ &\quad \{n + \alpha_2 m + a_2\}_q \{m + \beta_2 n + b_2\}_q \{n + \gamma_2(m + 1) + c_2\}_q. \end{aligned} \quad (86)$$

Pour la preuve suivante, comparez avec [3, p. 786].

Démonstration. Soit

$$f(x, y) = \sum_{m, n=0}^{\infty} A_{mn} x^m y^n. \quad (87)$$

On calcule d'abord les formules d'opérateur suivantes.

$$\{\theta_{q,1} + \gamma_1 \theta_{q,2} + c_1\}_q D_{q,x} f = \sum_{m, n=0}^{\infty} a_{m+1, n} \frac{\{m + \gamma_1 n + c_1\}_q}{\langle 1; q \rangle_m \langle 1; q \rangle_n} x^m y^n \quad (88)$$

$$\begin{aligned} &\{\theta_{q,1} + \alpha_1 \theta_{q,2} + a_1\}_q \{\theta_{q,1} + \beta_1 \theta_{q,2} + b_1\}_q f \\ &= \sum_{m, n=0}^{\infty} a_{mn} \frac{\{m + \alpha_1 n + a_1\}_q \{m + \beta_1 n + b_1\}_q}{\langle 1; q \rangle_m \langle 1; q \rangle_n} x^m y^n, \end{aligned} \quad (89)$$

où

$$A_{mn} \equiv \frac{a_{mn}}{\langle 1; q \rangle_m \langle 1; q \rangle_n}. \quad (90)$$

La première équation (84) est satisfaite quand

$$\frac{a_{m+1, n}}{a_{mn}} = \frac{\{m + \alpha_1 n + a_1\}_q \{m + \beta_1 n + b_1\}_q}{\{m + \gamma_1 n + c_1\}_q}, \quad (91)$$

La deuxième équation (84) est satisfaite quand

$$\frac{a_{m, n+1}}{a_{mn}} = \frac{\{n + \alpha_2 m + a_2\}_q \{n + \beta_2 m + b_2\}_q}{\{n + \gamma_2 m + c_2\}_q}. \quad (92)$$

En utilisant la notation de Horn, on a

$$f(m, n) \equiv \frac{A_{m+1, n}}{A_{mn}} = \frac{\{m + \alpha_1 n + a_1\}_q \{m + \beta_1 n + b_1\}_q}{\{m + \gamma_1 n + c_1\}_q \{m + 1\}_q}, \quad (93)$$

$$g(m, n) \equiv \frac{A_{m, n+1}}{A_{mn}} = \frac{\{n + \alpha_2 m + a_2\}_q \{n + \beta_2 m + b_2\}_q}{\{n + \gamma_2 m + c_2\}_q \{n + 1\}_q}. \quad (94)$$

Maintenant (85) découle de la condition de compatibilité

$$f(m, n)g(m + 1, n) = f(m, n + 1)g(m, n). \quad (95)$$

□

De même, nous trouvons que les fonctions q -hypergéométriques définies De (31)–(40), après remise à l'échelle, satisfaire les systèmes (84).

Cas premier. Supposons plutôt que

$$z = \sum_{m, n=0}^{\infty} C_{mn} x^{m+\rho} y^{n+\sigma}, \quad (96)$$

où ρ et σ sont des constantes réelles inconnues. Dans le cas précédent, C_{mn} devient A_{mn} . Nous avons maintenant les formules de récurrence

$$\begin{cases} F'(m + \rho, n + \sigma) C_{m+1, n} = F(m + \rho, n + \sigma) C_{m, n}, \\ G'(m + \rho, n + \sigma) C_{m, n+1} = G(m + \rho, n + \sigma) C_{m, n}, \end{cases} \quad (97)$$

qui découlent des formules de récurrence précédentes pour A_{mn} .

En comparant les coefficients de

$$x^{m+\rho} y^{n+\sigma}, \quad m = -1; n \geq 0 \quad (98)$$

à la première récurrence, et les coefficients de

$$x^{m+\rho} y^{n+\sigma}, \quad m \geq 0, n = -1 \quad (99)$$

dans la seconde récurrence, on obtient les équations [2, p. 27], [12, p.388] pour la détermination des exposants ρ et σ

$$\begin{cases} F'(\rho - 1, \sigma) = 0, \\ G'(\rho, \sigma - 1) = 0. \end{cases} \quad (100)$$

Cas deuxième. Pour la détermination des solutions au voisinage du point (∞, ∞) , on regarde les séries de la forme

$$z = \sum_{m,n=0}^{\infty} C_{mn} x^{\rho-m} y^{\sigma-n}, \quad (101)$$

où ρ et σ sont des constantes réelles inconnues. Nous avons maintenant les formules de récurrence

$$\begin{cases} F(\rho - m - 1, \sigma - n) C_{m+1,n} = F'(\rho - m - 1, \sigma - n) C_{m,n} \\ G(\rho - m, \sigma - n - 1) C_{m,n+1} = G'(\rho - m, \sigma - n - 1) C_{m,n}. \end{cases} \quad (102)$$

En comparant les coefficients de

$$x^{\rho-m} y^{\sigma-n}, \quad (103)$$

on obtient les équations [2, p. 28], [12, p.388] pour la détermination des exposants ρ et σ

$$\begin{cases} F(\rho, \sigma) = 0, \\ G(\rho, \sigma) = 0 \end{cases} \quad (104)$$

Cas troisième.

Pour la détermination des solutions au voisinage du point $(0, \infty)$, on regarde les séries de la forme

$$z = \sum_{m,n=0}^{\infty} C_{mn} x^{\rho+m} y^{\sigma-n}, \quad (105)$$

qui conduit à la formules de récurrence

$$\begin{cases} F'(\rho + m, \sigma - n) C_{m+1,n} = F(\rho + m, \sigma - n) C_{m,n} \\ G(\rho + m, \sigma - n - 1) C_{m,n+1} = G'(\rho + m, \sigma - n - 1) C_{m,n}. \end{cases} \quad (106)$$

En comparant les coefficients de

$$x^{\rho+m} y^{\sigma-n}, \quad (107)$$

on obtient les équations [2, p. 29], [12, p. 388] pour la détermination des exposants ρ et σ

$$\begin{cases} F'(\rho - 1, \sigma) = 0, \\ G(\rho, \sigma) = 0. \end{cases} \quad (108)$$

Cas quatrième.

Pour terminer, pour la détermination des solutions au voisinage du point $(\infty, 0)$, on regarde les séries de la forme

$$z = \sum_{m,n=0}^{\infty} C_{mn} x^{\rho-m} y^{\sigma+n}, \quad (109)$$

qui conduit à la formules de récurrence

$$\begin{cases} F(\rho - m - 1, \sigma + n) C_{m+1,n} = F'(\rho - m - 1, \sigma + n) C_{m,n} \\ G'(\rho - m, \sigma + n) C_{m,n+1} = G(\rho - m, \sigma + n) C_{m,n}. \end{cases} \quad (110)$$

En comparant les coefficients de

$$x^{\rho-m} y^{\sigma+n}, \quad (111)$$

on obtient les équations [2, p. 29], [12, p.388] pour la détermination des exposants ρ et σ

$$\begin{cases} F(\rho, \sigma) = 0, \\ G'(\rho, \sigma - 1) = 0. \end{cases} \quad (112)$$

3. LES FONCTIONS CONFLUENTES DE q -APPELL

Le but de cette partie est de trouver les équations aux q -différences et les q -intégrales pour les fonctions q -hypergéométriques doubles confluentes.

Le mode de preuve est très simple, laissez le paramètre manquant tendre vers ∞ . Cette courte section constitue la base des autres solutions aux systèmes des équations aux q -différences et leurs intégrales q -Euler pour les fonctions q -Appell confluentes dans la section 4.

Théorème 3.1. *Équations aux q -différences pour Υ_1 .*

$$\begin{cases} \{ \{ \theta_{q,1} + \theta_{q,2} + c \}_q D_{q,x} - \{ \theta_{q,1} + \theta_{q,2} + a \}_q \{ \theta_{q,1} + b \}_q \} f(x, y; q) = 0, \\ \{ \{ \theta_{q,1} + \theta_{q,2} + c \}_q (1 - q) D_{q,y} - \{ \theta_{q,1} + \theta_{q,2} + a \}_q \} f(x, y; q) = 0. \end{cases} \quad (113)$$

Théorème 3.2. *Une q -intégrale pour Υ_1 . Un q -analogue de [4, 4.5].*

$$\Upsilon_1(a; b; c | q; x, y) \cong \Gamma_q \left[\begin{matrix} c \\ a, c - a \end{matrix} \right] \int_0^1 t^{a-1} e_q(yt) \frac{(qt; q)_{c-a-1}}{(xt; q)_b} d_q(t). \quad (114)$$

Démonstration. Soit $b' \rightarrow \infty$ dans [7, 10.104]. □

Théorème 3.3. *Équation aux q -différences pour Υ_2 .*

$$\begin{cases} [\{\theta_{q,1} + \theta_{q,2} + c\}_q(1-q)D_{q,x} - \{\theta_{q,1} + b_1\}_q] f(x, y; q) = 0, \\ [\{\theta_{q,1} + \theta_{q,2} + c\}_q(1-q)D_{q,y} - \{\theta_{q,2} + b_2\}_q] f(x, y; q) = 0. \end{cases} \quad (115)$$

Théorème 3.4. *Une q -intégrale pour Υ_2 . Un q -analogue de [4, 4.23].*

$$\begin{aligned} \Upsilon_2(b_1; b_2; c|q; x, y) &\cong \Gamma_q \left[\begin{matrix} c \\ b_1, c - b_1 \end{matrix} \right] \int_0^1 t^{b_1-1} (qt; q)_{c-b_1-1} e_q(xt) \\ &\sum_{m=0}^{\infty} \frac{\langle b_2; q \rangle_m}{\langle 1, c - b_1; q \rangle_m} y^m (tq^{c-b_1}; q)_m d_q(t). \end{aligned} \quad (116)$$

Démonstration. Soit $a \rightarrow \infty$ dans Φ_1 . □

Théorème 3.5. *Équation aux q -différences pour Υ_3 .*

$$\begin{cases} [\{\theta_{q,1} + \theta_{q,2} + c\}_q(1-q)D_{q,x} - \{\theta_{q,1} + b\}_q] f(x, y; q) = 0, \\ [\{\theta_{q,1} + \theta_{q,2} + c\}_q(1-q)^2 D_{q,y} - I] f(x, y; q) = 0. \end{cases} \quad (117)$$

Théorème 3.6. *Une q -intégrale pour Υ_3 . Un q -analogue de [4, 4.29].*

$$\begin{aligned} \Upsilon_3(b; c|q; x, y) &\cong \Gamma_q \left[\begin{matrix} c \\ b, c - b \end{matrix} \right] \int_0^1 t^{b-1} (qt; q)_{c-b-1} e_q(xt) \\ &\sum_{m=0}^{\infty} \frac{y^m}{\langle 1, c - b; q \rangle_m} (tq^{c-b}; q)_m d_q(t). \end{aligned} \quad (118)$$

Démonstration. Soit $b_2 \rightarrow \infty$ dans (116). □

Théorème 3.7. *Équation aux q -différences pour $\Psi_1(a; b; c_1, c_2|q; x, y)$.*

$$\begin{cases} [\{\theta_{q,1} + c_1\}_q D_{q,x} - \{\theta_{q,1} + \theta_{q,2} + a\}_q \{\theta_{q,1} + b\}_q] f(x, y; q) = 0, \\ [\{\theta_{q,2} + c_2\}_q(1-q)D_{q,y} - \{\theta_{q,1} + \theta_{q,2} + a\}_q] f(x, y; q) = 0. \end{cases} \quad (119)$$

Théorème 3.8. *Équation aux q -différences pour $\Psi_2(a; c_1, c_2|q; x, y)$.*

$$\begin{cases} [\{\theta_{q,1} + c_1\}_q(1-q)D_{q,x} - \{\theta_{q,1} + \theta_{q,2} + a\}_q] f(x, y; q) = 0, \\ [\{\theta_{q,2} + c_2\}_q(1-q)D_{q,y} - \{\theta_{q,1} + \theta_{q,2} + a\}_q] f(x, y; q) = 0. \end{cases} \quad (120)$$

Théorème 3.9. *Équation aux q -différences pour $\Psi_3(a; c|q; x, y)$.*

$$\begin{cases} [\{\theta_{q,1} + c\}_q(1-q)D_{q,x} - \{\theta_{q,1} + \theta_{q,2} + a\}_q] f(x, y; q) = 0, \\ [D_{q,y} - \{\theta_{q,1} + \theta_{q,2} + a\}_q] f(x, y; q) = 0. \end{cases} \quad (121)$$

Théorème 3.10. *Équation aux q -différences pour $\Xi_1(a_1, a_2; b; c|q; x, y)$.*

$$\begin{cases} [\{\theta_{q,1} + \theta_{q,2} + c\}_q D_{q,x} - \{\theta_{q,1} + a_1\}_q \{\theta_{q,1} + b\}_q] f(x, y; q) = 0, \\ [\{\theta_{q,1} + \theta_{q,2} + c\}_q (1-q)D_{q,y} - \{\theta_{q,1} + a_2\}_q] f(x, y; q) = 0. \end{cases} \quad (122)$$

Théorème 3.11. *Équation aux q -différences pour $\Xi_2(a; b; c|q; x, y)$.*

$$\begin{cases} [\{\theta_{q,1} + \theta_{q,2} + c\}_q D_{q,x} - \{\theta_{q,1} + a\}_q \{\theta_{q,1} + b\}_q] f(x, y; q) = 0, \\ [\{\theta_{q,1} + \theta_{q,2} + c\}_q (1-q)^2 D_{q,y} - I] f(x, y; q) = 0. \end{cases} \quad (123)$$

Théorème 3.12. *Équation aux q -différences pour ${}_2\phi_1(a_1, \infty; c_1|q; x) {}_2\phi_1(a_2, \infty; c_2|q; y)$.*

$$\begin{cases} [\{\theta_{q,1} + c_1\}_q (1-q)D_{q,x} - \{\theta_{q,1} + a_1\}_q] f(x, y; q) = 0, \\ [\{\theta_{q,2} + c_2\}_q (1-q)D_{q,y} - \{\theta_{q,2} + a_2\}_q] f(x, y; q) = 0. \end{cases} \quad (124)$$

Théorème 3.13. *Équation aux q -différences pour ${}_2\phi_1(a, b; c_1|q; x) {}_2\phi_1(2\infty; c_2|q; y)$.*

$$\begin{cases} [\{\theta_{q,1} + c_1\}_q D_{q,x} - \{\theta_{q,1} + a\}_q \{\theta_{q,1} + b\}_q] f(x, y; q) = 0, \\ [\{\theta_{q,2} + c_2\}_q (1-q)^2 D_{q,y} - I] f(x, y; q) = 0. \end{cases} \quad (125)$$

4. D'AUTRES SOLUTIONS ET LEURS INTÉGRALES DE q -EULER POUR LES FONCTIONS CONFLUENTES DE q -APPELL

4.1. **D'autres solutions des équations de fonction Υ_1 .** La seule fonction confluyente avec Φ_1 est Υ_1 . En nous appuyant sur les résultats dans [10], nous énonçons simplement les résultats pour Υ_1 .

Théorème 4.1. *Les autres solutions de (113) au voisinage de $(0, 0)$ sont les (fonctions confluyente du q -Horn):*

$$\Upsilon_{11}(a, b, c; x, y; q) \equiv x^{1-c} \sum_{m,n=0}^{\infty} (-1)^{m-n} \text{QE} \left(-\binom{m-n}{2} + (m-n)(c-2) \right) \frac{\langle a+1-c; q \rangle_m \langle b+1-c; q \rangle_{m-n} \langle c-1; q \rangle_{n-m}}{\langle 1; q \rangle_m \langle 1; q \rangle_n} x^{m-n} y^n, \quad (126)$$

et

$$\Upsilon_{12}(a, b, c; x, y; q) = x^{1-c} \sum_{m,n=0}^{\infty} \frac{\langle a+1-c; q \rangle_{m+n} \langle b+1-c \rangle_m}{\langle 2-c \rangle_m \langle 1; q \rangle_{m+n} \langle 1; q \rangle_n} x^m y^n. \quad (127)$$

Démonstration. Soit $b_2 \rightarrow \infty$ dans [10]. $\square$

Passons maintenant aux représentations q -intégrales des solutions du système pour Υ_1 . Le formulaire opérateur (82) donne l'intégrale q pour le premier $\Upsilon_{11}(x, y; q)$.

Théorème 4.2.

$$\Upsilon_{11}(a, b, c; x, y; q) \cong x^{1-c} \Gamma_q \left[\begin{matrix} 2-c \\ a+1-c, 1-a \end{matrix} \right] \int_0^1 t^{a-c} \frac{(qt; q)_{-a}}{(xt; q)_{b+1-c}} {}_3\phi_2 \left[\begin{matrix} a, \infty \\ c-b \end{matrix} \middle| q; yq^{b-c+a} \right] \frac{((xt)^{-1}q^{c-b}; q)_k}{(t^{-1}q^a; q)_k} d_q(t). \quad (128)$$

Démonstration.

$$\Upsilon_{11}(a, b, c; x, y; q) \cong x^{1-c} \Gamma_q \left[\begin{matrix} 2-c-\theta_{q,2} \\ a+1-c, 1-a-\theta_{q,2} \end{matrix} \right] \int_0^1 t^{a-c} \frac{(qt; q)_{-a-\theta_{q,2}}}{(xt; q)_{b_1+1-c-\theta_{q,2}}} {}_2\phi_1 \left[\begin{matrix} c-1, \infty \\ c-b_1 \end{matrix} \middle| q; \frac{y}{x} \right] d_q(t) \stackrel{\text{par(77)}}{=} \text{LCD}. \quad (129)$$

$\square$

De même, par permutation des paramètres on obtient

Théorème 4.3.

$$\Upsilon_{11}(a, b, c; x, y; q) \cong x^{1-c} \Gamma_q \left[\begin{matrix} 2-c \\ b+1-c, 1-b \end{matrix} \right] \int_0^1 t^{b-c} \frac{(qt; q)_{-b}}{(xt; q)_{a+1-c}} e_q \left(\frac{y}{x} q^{b-1} \right) d_q(t). \quad (130)$$

Les solutions au voisinage de $(0, \infty)$ n'ont aucun sens pour Υ_1 car il n'y a pas assez de paramètres avec m_2 dans la série. Nous nous tournons donc vers des solutions au voisinage de $(\infty, 0)$. D'Ansatz IV on obtient les équations

$$\begin{cases} \{\sigma\}_q \{c + \sigma + \rho - 1\}_q = 0, \\ \{b_1 + \rho\}_q \{a + \rho + \sigma\}_q = 0. \end{cases} \quad (131)$$

Cela a les trois solutions

$$\begin{aligned} \sigma &= 0, \quad \rho = -b_1; \\ \sigma &= 0, \quad \rho = -a; \\ \sigma &= b_1 + 1 - c, \quad \rho = -b_1. \end{aligned} \quad (132)$$

Cela donne la solution

$$\begin{aligned} g_1(x, y; q) &= x^{-b_1} \sum_{m, n=0}^{\infty} \frac{\langle b_1 + 1 - c, b_1; q \rangle_n \langle a - b_1 - n; q \rangle_m}{\langle 1, c - b_1 - n; q \rangle_m \langle 1, b_1 + 1 - a; q \rangle_n} y^m x^{-n} q^{n(c-a-b_1)} \\ &= x^{-b_1} \sum_{m, n=0}^{\infty} \frac{\langle a - b_1; q \rangle_{m-n} \langle b_1; q \rangle_n \langle b_1 + 1 - c; q \rangle_{n-m}}{\langle 1; q \rangle_m \langle 1; q \rangle_n} \\ &\quad (-y)^m (-x)^{-n} \text{QE} \left(-\binom{m}{2} - \binom{n}{2} + n(c - 2b_1 - 1) - m(c - b_1 - n) \right). \end{aligned} \quad (133)$$

Nous pouvons à nouveau réécrire ceci sous la forme d'une fonction confluente du q -Horn.

La deuxième solution est

$$g_4(x, y; q) = x^{-a} \Upsilon_1 \left(a; a + 1 - c; a + 1 - b_1 \middle| q; \frac{1}{x}, -\frac{y}{x} q^{a+1-c} \right). \quad (134)$$

Théorème 4.4. *La troisième solution est*

$$h_3(x, y; q) = x^{-a} \sum_{m, n=0}^{\infty} \frac{\langle a+1-c; q \rangle_{n-m} \langle a; q \rangle_n}{\langle a+1-b_1; q \rangle_n \langle 1; q \rangle_m \langle 1; q \rangle_{n-m}} x^{-n} y^m \quad (135)$$

$$\text{QE}((c-a)(m-n) + (1-b_1)n).$$

Théorème 4.5. *La quatrième solution est*

$$h_4(x, y; q) = x^{-b_1} y^{b_1+1-c} \sum_{m, n=0}^{\infty} \frac{\langle a+1-c; q \rangle_{m-n} \langle b_1; q \rangle_n}{\langle 2+b_1-c; q \rangle_m \langle 1; q \rangle_n \langle 1; q \rangle_{m-n}} y^m x^{-n} \text{QE}(-nb_1+n). \quad (136)$$

4.2. D'autres solutions des équations de fonction Ψ_1 .

Théorème 4.6. *L'équation aux q -différence (119) a les quatre solutions indépendantes suivantes au voisinage de $(0, 0)$.*

$$\begin{aligned} f_1(x, y; q) &\equiv \Psi_1(a; b; c_1, c_2 | q; x, y), \\ f_2(x, y; q) &\equiv x^{1-c_1} \Psi_1(a-c_1+1; b-c_1+1; 2-c_1, c_2 | q; x, y), \\ f_3(x, y; q) &\equiv y^{1-c_2} \Psi_1(a-c_2+1; b; c_1, 2-c_2 | q; x, y), \\ f_4(x, y; q) &\equiv x^{1-c_1} y^{1-c_2} \\ &\times \Psi_1(a-c_1-c_2+2; b-c_1+1; 2-c_1, 2-c_2 | q; x, y). \end{aligned} \quad (137)$$

Théorème 4.7. *Une représentation q -intégrale de $\Psi_1(a; b; c_1, c_2 | q; x, y)$.*

$$\begin{aligned} \Psi_1(a; b; c_1, c_2 | q; x, y) &\cong \Gamma_q \left[\begin{matrix} c_1 \\ a, c_1 - a \end{matrix} \right] \int_0^1 t^{a-1} \frac{(qt; q)_{c_1-a-1}}{(xt; q)_{b_1}} \\ &\sum_{m=0}^{\infty} \frac{\langle a+1-c_1; q \rangle_m}{\langle 1, c_2; q \rangle_m} (-ty)^m (tq^{c_1-a}; q)_{-m} \\ &\text{QE} \left(-\binom{m}{2} + m(c_1-a-1) \right) d_q(t). \end{aligned} \quad (138)$$

Théorème 4.8. *Les fonctions $f_3(x, y; q)$ et $f_4(x, y; q)$ ci-dessus ont des représentations q -intégrales*

$$\begin{aligned}
 f_3(x, y; q) &\cong x^{1-c_1} \Gamma_q \left[\begin{matrix} c_2 \\ a, c_2 - a \end{matrix} \right] \int_0^1 t^{a-1} e_q(yt) (qt; q)_{c_2-a-1} (yt; q)_{b_2} \\
 &\sum_{m=0}^{\infty} \frac{\langle a+1-c_1, a+1-c_2, b_1+1-c_1; q \rangle_m}{\langle 1, 2-c_1, a; q \rangle_m} (-xt)^m (tq^{c_2-a}; q)_{-m} \\
 &\text{QE} \left(-\binom{m}{2} + m(c_2 - a - 1) \right) d_q(t).
 \end{aligned} \tag{139}$$

$$\begin{aligned}
 f_4(x, y; q) &\cong x^{1-c_1} y^{1-c_2} \Gamma_q \left[\begin{matrix} 2-c_1 \\ a+1-c_1, 1-a \end{matrix} \right] \int_0^1 t^{a-c_1} \frac{(qt; q)_{-a}}{(xt; q)_{b_1+1-c_1}} \\
 &\sum_{m=0}^{\infty} \frac{\langle a+2-c_1-c_2, a; q \rangle_m}{\langle 1, 2-c_2, a+1-c_1; q \rangle_m} (-yt)^m (tq^{1-a}; q)_{-m} \\
 &\text{QE} \left(-\binom{m}{2} - ma \right) d_q(t).
 \end{aligned} \tag{140}$$

Théorème 4.9. *L'équation aux q -différence (120) a les quatre solutions indépendantes suivantes aux voisinage de $(0, 0)$.*

$$\begin{aligned}
 f_1(x, y; q) &\equiv \Psi_2(a; c_1, c_2 | q; x, y), \\
 f_2(x, y; q) &\equiv x^{1-c_1} \Psi_2(a - c_1 + 1; 2 - c_1, c_2 | q; x, y), \\
 f_3(x, y; q) &\equiv y^{1-c_2} \Psi_2(a - c_2 + 1; c_1, 2 - c_2 | q; x, y), \\
 f_4(x, y; q) &\equiv x^{1-c_1} y^{1-c_2} \\
 &\times \Psi_2(a - c_1 - c_2 + 2; 2 - c_1, 2 - c_2 | q; x, y).
 \end{aligned} \tag{141}$$

Théorème 4.10. *Une représentation q -intégrale de $\Psi_2(a; c_1, c_2|q; x, y)$*

$$\begin{aligned} \Psi_2(a; b_1, b_2; c_1, c_2|q; x, y) &\cong \Gamma_q \left[\begin{matrix} c_1 \\ a, c_1 - a \end{matrix} \right] \int_0^1 t^{a-1} e_q(xt) (qt; q)_{c_1-a-1} \\ &\sum_{m=0}^{\infty} \frac{\langle a+1-c_1; q \rangle_m}{\langle 1, c_2; q \rangle_m} (-ty)^m (tq^{c_1-a}; q)_{-m} \\ &\text{QE} \left(-\binom{m}{2} + m(c_1 - a - 1) \right) d_q(t). \end{aligned} \quad (142)$$

Théorème 4.11. *Les fonctions $f_3(x, y; q)$ et $f_4(x, y; q)$ ci-dessus ont des représentations q -intégrales*

$$\begin{aligned} f_3(x, y; q) &\cong x^{1-c_1} \Gamma_q \left[\begin{matrix} c_2 \\ a, c_2 - a \end{matrix} \right] \int_0^1 t^{a-1} e_q(yt) (qt; q)_{c_2-a-1} \\ &\sum_{m=0}^{\infty} \frac{\langle a+1-c_1, a+1-c_2; q \rangle_m}{\langle 1, 2-c_1, a; q \rangle_m} (-xt)^m (tq^{c_2-a}; q)_{-m} \\ &\text{QE} \left(-\binom{m}{2} + m(c_2 - a - 1) \right) d_q(t). \end{aligned} \quad (143)$$

$$\begin{aligned} f_4(x, y; q) &\cong x^{1-c_1} y^{1-c_2} \Gamma_q \left[\begin{matrix} 2-c_1 \\ a+1-c_1, 1-a \end{matrix} \right] \int_0^1 t^{a-c_1} e_q(xt) (qt; q)_{-a} \\ &\sum_{m=0}^{\infty} \frac{\langle a+2-c_1-c_2, a; q \rangle_m}{\langle 1, 2-c_2, a+1-c_1; q \rangle_m} (-yt)^m (tq^{1-a}; q)_{-m} \\ &\text{QE} \left(-\binom{m}{2} - ma \right) d_q(t). \end{aligned} \quad (144)$$

Théorème 4.12. *L'équation aux q -différence (121) a les trois solutions suivantes indépendants aux voisinage de $(0, 0)$:*

$$\begin{aligned} f_1(x, y; q) &\equiv \Psi_3(a; c|q; x, y), \\ f_2(x, y; q) &\equiv x^{1-c} \Psi_3(a-c+1; 2-c|q; x, y), \\ f_3(x, y; q) &\equiv y^{1-c} \Psi_3(a-c+1; 2-c|q; y, x). \end{aligned} \quad (145)$$

Théorème 4.13. Une représentation q -intégrale de $\Psi_3(a; c|q; x, y)$

$$\begin{aligned} \Psi_3(a; c|q; x, y) &\cong \Gamma_q \left[\begin{matrix} c_1 \\ a, c_1 - a \end{matrix} \right] \int_0^1 t^{a-1} e_q(xt) (qt; q)_{c_1-a-1} \\ &\sum_{m=0}^{\infty} \frac{\langle a+1-c_1; q \rangle_m}{\langle 1; q \rangle_m} (-ty)^m (tq^{c_1-a}; q)_{-m} \times \\ &\text{QE} \left(-\binom{m}{2} + m(c_1 - a - 1) \right) d_q(t). \end{aligned} \quad (146)$$

4.3. Une représentation q -intégrale des fonctions Ξ_q . En laissant les paramètres tendre vers ∞ dans la représentation q -intégrale pour $\Phi_{3,q}$ on obtient les résultats suivantes :

Théorème 4.14. La fonction Ξ_1 a une représentation q -intégrale

$$\begin{aligned} \Xi_1(a_1, a_2; b; c|q; x, y) &\cong \Gamma_q \left[\begin{matrix} c \\ a_1, c - a_1 \end{matrix} \right] \int_0^1 t^{a_1-1} \frac{(qt; q)_{c-a_1-1}}{(xt; q)_b} \\ &\sum_{m=0}^{\infty} \frac{\langle a_2; q \rangle_m}{\langle 1, c - a_1; q \rangle_m} y^m (q^{c-a_1}t; q)_m d_q(t). \end{aligned} \quad (147)$$

Théorème 4.15. La fonction Ξ_2 a une représentation q -intégrale

$$\begin{aligned} \Xi_2(a; b; c|q; x, y) &\cong \Gamma_q \left[\begin{matrix} c \\ a, c - a \end{matrix} \right] \int_0^1 t^{a-1} \frac{(qt; q)_{c-a-1}}{(xt; q)_b} \\ &\sum_{m=0}^{\infty} \frac{1}{\langle 1, c - a; q \rangle_m} y^m (q^{c-a}t; q)_m d_q(t). \end{aligned} \quad (148)$$

5. LES ÉQUATIONS DIFFÉRENTIELLES PARTIELLES DES FONCTIONS DE HORN

Les opérateurs opèrent sur la fonction Horn respectivement à partir de la gauche et D désigne la dérivée. θ_i désigne l'opérateur d'Euler $x_i D_i$. Comparer avec Yoshida [14, p. 331].

(1) Système $G_1(a; b_1; b_2; x, y)$

$$\begin{cases} (-\theta_1 + \theta_2 + b_1 - 1)D_x - (\theta_1 + \theta_2 + a)(\theta_1 - \theta_2 + b_2) = 0, \\ (\theta_1 - \theta_2 + b_2 - 1)D_y - (\theta_1 + \theta_2 + a)(-\theta_1 + \theta_2 + b_1) = 0. \end{cases} \quad (149)$$

(2) Système $G_2(a_1; a_2; b_1, b_2; x, y)$

$$\begin{cases} (-\theta_1 + \theta_2 + b_1 - 1)D_x - (\theta_1 + a_1)(\theta_1 - \theta_2 + b_2) = 0, \\ (\theta_1 - \theta_2 + b_2 - 1)D_y - (\theta_2 + a_2)(-\theta_1 + \theta_2 + b_1) = 0. \end{cases} \quad (150)$$

(3) Système $G_3(a_1; a_2; x, y)$

$$\begin{cases} (-\theta_1 + 2\theta_2 + a_1 - 1)D_x - (2\theta_1 - \theta_2 + a_2)(2\theta_1 - \theta_2 + a_2 + 1) = 0, \\ (2\theta_1 - \theta_2 + a_2 - 1)D_y - (-\theta_1 + 2\theta_2 + a_1)(-\theta_1 + 2\theta_2 + a_1 + 1) = 0. \end{cases} \quad (151)$$

(4) Système $H_1(a; b; c; d; x, y)$

$$\begin{cases} (\theta_1 + d)D_x - (\theta_1 - \theta_2 + a)(\theta_1 + \theta_2 + b) = 0, \\ (\theta_1 - \theta_2 + a - 1)D_y - (\theta_1 + \theta_2 + b)(\theta_2 + c) = 0. \end{cases} \quad (152)$$

(5) Système $H_2(a; b; c; d; e; x, y)$

$$\begin{cases} (\theta_1 + e)D_x - (\theta_1 - \theta_2 + a)(\theta_1 + b) = 0, \\ (\theta_1 - \theta_2 + a - 1)D_y - (\theta_2 + c)(\theta_2 + d) = 0. \end{cases} \quad (153)$$

(6) Système $H_3(a; b; c; x, y)$

$$\begin{cases} (\theta_1 + \theta_2 + c)D_x - (2\theta_1 + \theta_2 + a)(2\theta_1 + \theta_2 + a + 1) = 0, \\ (\theta_1 + \theta_2 + c)D_y - (2\theta_1 + \theta_2 + a)(\theta_2 + b) = 0. \end{cases} \quad (154)$$

(7) Système $H_4(a; b; c; d; x, y)$

$$\begin{cases} (\theta_1 + c)D_x - (2\theta_1 + \theta_2 + a)(2\theta_1 + \theta_2 + a + 1) = 0, \\ (\theta_2 + d)D_y - (2\theta_1 + \theta_2 + a)(\theta_2 + b) = 0. \end{cases} \quad (155)$$

(8) Système $H_5(a; b; c; x, y)$

$$\begin{cases} (-\theta_1 + \theta_2 + b - 1)D_x - (2\theta_1 + \theta_2 + a)(2\theta_1 + \theta_2 + a + 1) = 0, \\ (\theta_2 + c)D_y - (2\theta_1 + \theta_2 + a)(-\theta_1 + \theta_2 + b) = 0. \end{cases} \quad (156)$$

(9) Système $H_6(a; b; c; x, y)$

$$\begin{cases} (-\theta_1 + \theta_2 + b - 1)D_x - (2\theta_1 - \theta_2 + a)(2\theta_1 - \theta_2 + a + 1) = 0, \\ (2\theta_1 - \theta_2 + a - 1)D_y - (-\theta_1 + \theta_2 + b)(\theta_2 + c) = 0. \end{cases} \quad (157)$$

(10) Système $H_7(a; b; c; d; x, y)$

$$\begin{cases} (\theta_1 + d)D_x - (2\theta_1 - \theta_2 + a)(2\theta_1 - \theta_2 + a + 1) = 0, \\ (2\theta_1 - \theta_2 + a - 1)D_y - (\theta_2 + b)(\theta_2 + c) = 0. \end{cases} \quad (158)$$

Théorème 5.1. *Une représentation intégrale de $G_1(a; b_1; b_2; -x, -y)$. Une version corrigée de [3, (9.1.3) p. 809].*

$$G_1(a; b_1; b_2; -x, -y) = \Gamma \left[\begin{matrix} 1 - b_1 \\ b_2, 1 - b_1 - b_2 \end{matrix} \right] \int_0^1 t^{b_2-1} (1-t)^{-b_1-b_2} (1-xt)^{-a} \left(1 - \frac{y}{t(1-xt)} \right)^{-a} dt. \quad (159)$$

Une version corrigée de [3, p. 819].

$$H_1(a; b; c; d; x, -y) = {}_2F_1(a - \theta_2, b + \theta_2; d; x) {}_2F_1(b, c; 1 - a; y). \quad (160)$$

Cela implique

Théorème 5.2. *Une représentation intégrale de $H_1(a; b; c; d; x, -y)$. Une version corrigée de [3, p. 819].*

$$H_1(a; b; c; d; x, -y) = \Gamma \left[\begin{matrix} d \\ b, d - b \end{matrix} \right] \int_0^1 t^{b-1} (1-t)^{d-b-1} (1-xt)^{-a} {}_2F_1 \left(c, 1 - d + b; 1 - a; \frac{-ty(1-xt)}{1-t} \right) dt. \quad (161)$$

6. LES ÉQUATIONS AUX q -DIFFÉRENCE DES FONCTIONS DE q -HORN (NORMALE)

Les systèmes suivantes d'équations aux différences q pour les fonctions q -Horn normales sont prouvées à la main par comparaison avec des coefficients correspondants. Notez que les trois fonctions G q -Horn symétriques ont des équations de différence q symétriques. La fonction $H_{2,q}$ est également symétrique par rapport à c et d .

(1) Système $G_1(a; b_1; b_2|q; x, y)$

$$\begin{cases} \{-\theta_{q,1} + \theta_{q,2} + b_1 - 1\}_q D_{q,x} - \{\theta_{q,1} + \theta_{q,2} + a\}_q \{\theta_{q,1} - \theta_{q,2} + b_2\}_q = 0, \\ \{\theta_{q,1} - \theta_{q,2} + b_2 - 1\}_q D_{q,y} - \{\theta_{q,1} + \theta_{q,2} + a\}_q \{-\theta_{q,1} + \theta_{q,2} + b_1\}_q = 0. \end{cases} \quad (162)$$

(2) Système $G_2(a_1; a_2; b_1, b_2|q; x, y)$

$$\begin{cases} \{-\theta_{q,1} + \theta_{q,2} + b_1 - 1\}_q D_{q,x} - \{\theta_{q,1} + a_1\}_q \{\theta_{q,1} - \theta_{q,2} + b_2\}_q = 0, \\ \{\theta_{q,1} - \theta_{q,2} + b_2 - 1\}_q D_{q,y} - \{\theta_{q,2} + a_2\}_q \{-\theta_{q,1} + \theta_{q,2} + b_1\}_q = 0. \end{cases} \quad (163)$$

(3) Système $G_3(a_1; a_2|q; x, y)$

$$\begin{cases} \{-\theta_{q,1} + 2\theta_{q,2} + a_1 - 1\}_q D_{q,x} - \{2\theta_{q,1} - \theta_{q,2} + a_2\}_q \{2\theta_{q,1} - \theta_{q,2} + a_2 + 1\}_q = 0, \\ \{2\theta_{q,1} - \theta_{q,2} + a_2 - 1\}_q D_{q,y} - \{-\theta_{q,1} + 2\theta_{q,2} + a_1\}_q \{-\theta_{q,1} + 2\theta_{q,2} + a_1 + 1\}_q = 0. \end{cases} \quad (164)$$

(4) Système $H_1(a; b; c; d|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + d\}_q D_{q,x} - \{\theta_{q,1} - \theta_{q,2} + a\}_q \{\theta_{q,1} + \theta_{q,2} + b\}_q = 0, \\ \{\theta_{q,1} - \theta_{q,2} + a - 1\}_q D_{q,y} - \{\theta_{q,1} + \theta_{q,2} + b\}_q \{\theta_{q,2} + c\}_q = 0. \end{cases} \quad (165)$$

(5) Système $H_2(a; b; c; d; e|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + e\}_q D_{q,x} - \{\theta_{q,1} - \theta_{q,2} + a\}_q \{\theta_{q,1} + b\}_q = 0, \\ \{\theta_{q,1} - \theta_{q,2} + a - 1\}_q D_{q,y} - \{\theta_{q,2} + c\}_q \{\theta_{q,2} + d\}_q = 0. \end{cases} \quad (166)$$

(6) Système $H_3(a; b; c|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + \theta_{q,2} + c\}_q D_{q,x} - \{2\theta_{q,1} + \theta_{q,2} + a\}_q \{2\theta_{q,1} + \theta_{q,2} + a + 1\}_q = 0, \\ \{\theta_{q,1} + \theta_{q,2} + c\}_q D_{q,y} - \{2\theta_{q,1} + \theta_{q,2} + a\}_q \{\theta_{q,2} + b\}_q = 0. \end{cases} \quad (167)$$

(7) Système $H_4(a; b; c; d|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + c\}_q D_{q,x} - \{2\theta_{q,1} + \theta_{q,2} + a\}_q \{2\theta_{q,1} + \theta_{q,2} + a + 1\}_q = 0, \\ \{\theta_{q,2} + d\}_q D_{q,y} - \{2\theta_{q,1} + \theta_{q,2} + a\}_q \{\theta_{q,2} + b\}_q = 0. \end{cases} \quad (168)$$

(8) Système $H_5(a; b; c|q; x, y)$

$$\begin{cases} \{-\theta_{q,1} + \theta_{q,2} + b - 1\}_q D_{q,x} - \{2\theta_{q,1} + \theta_{q,2} + a\}_q \{2\theta_{q,1} + \theta_{q,2} + a + 1\}_q = 0, \\ \{\theta_{q,2} + c\}_q D_{q,y} - \{2\theta_{q,1} + \theta_{q,2} + a\}_q \{-\theta_{q,1} + \theta_{q,2} + b\}_q = 0. \end{cases} \quad (169)$$

(9) Système $H_6(a; b; c|q; x, y)$

$$\begin{cases} \{-\theta_{q,1} + \theta_{q,2} + b - 1\}_q D_{q,x} - \{2\theta_{q,1} - \theta_{q,2} + a\}_q \{2\theta_{q,1} - \theta_{q,2} + a + 1\}_q = 0, \\ \{2\theta_{q,1} - \theta_{q,2} + a - 1\}_q D_{q,y} - \{-\theta_{q,1} + \theta_{q,2} + b\}_q \{\theta_{q,2} + c\}_q = 0. \end{cases} \quad (170)$$

(10) Système $H_7(a; b; c; d|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + d\}_q D_{q,x} - \{2\theta_{q,1} - \theta_{q,2} + a\}_q \{2\theta_{q,1} - \theta_{q,2} + a + 1\}_q = 0, \\ \{2\theta_{q,1} - \theta_{q,2} + a - 1\}_q D_{q,y} - \{\theta_{q,2} + b\}_q \{\theta_{q,2} + c\}_q = 0. \end{cases} \quad (171)$$

7. D'AUTRES SOLUTIONS DES ÉQUATIONS AUX q -DIFFÉRENCES DES FONCTIONS DE q -HORN

Le but de cette section est de trouver d'autres solutions des systèmes précédents d'équations aux différences q par la méthode de Horn. Seulement quelques exemples sont donnés.

7.1. Le système $G_{2,q}$. Solutions au voisinage de $(0, 0)$.

Du premier Ansatz on obtient les équations

$$\begin{cases} \{\rho\}_q \{b_1 + \sigma - \rho\}_q = 0, \\ \{\sigma\}_q \{b_2 + \rho - \sigma\}_q = 0. \end{cases} \quad (172)$$

Cela a les trois solutions

$$\begin{aligned} \rho &= 0, \quad \sigma = 0, \\ \rho &= b_1, \quad \sigma = 0, \\ \rho &= 0, \quad \sigma = b_2. \end{aligned} \quad (173)$$

La première solution est $G_{2,q}$.

Théorème 7.1. *La deuxième solution est*

$$\begin{aligned} g_2(x, y; q) &= x^{b_1} \sum_{m,n=0}^{\infty} \frac{\langle b_1 + b_2; q \rangle_{m-n} \langle a_2; q \rangle_n \langle a_1 + b_1; q \rangle_m}{\langle b_1 + 1; q \rangle_m \langle 1; q \rangle_n \langle 1; q \rangle_{m-n}} \\ &(-x)^m (-y)^n \text{QE} \left(\binom{m-n}{2} + m - n \right). \end{aligned} \quad (174)$$

Cette solution est une q -analogue de [12, p.391].

Démonstration. De la récurrence (97) on obtient

$$\begin{cases} \frac{C_{m+1,n}}{C_{mn}} = \frac{\langle a_1 + b_1 + m, b_1 + b_2 + m - n; q \rangle_1}{\langle b_1 + m + 1, n - m - 1; q \rangle_1}, \\ \frac{C_{m,n+1}}{C_{mn}} = \frac{\langle a_2 + n, n - m; q \rangle_1}{\langle n + 1, b_1 + b_2 + m - 1 - n; q \rangle_1}, \end{cases} \quad (175)$$

nous trouvons

$$g_2(x, y; q) = x^{b_1} \sum_{m,n=0}^{\infty} \frac{\langle b_1 + b_2; q \rangle_{m-n} \langle a_2; q \rangle_n \langle a_1 + b_1; q \rangle_m}{\langle b_1 + 1; q \rangle_m \langle 1; q \rangle_n \langle n - m; q \rangle_{m-n}} \quad (176)$$

$x^m y^n$ par [7, 6.14] LCD.

□

Théorème 7.2. *La troisième solution est*

$$g_3(x, y; q) = y^{b_2} \sum_{m,n=0}^{\infty} \frac{\langle b_1 + b_2; q \rangle_{n-m} \langle a_1; q \rangle_m \langle a_2 + b_2; q \rangle_n}{\langle b_2 + 1; q \rangle_n \langle 1; q \rangle_m \langle 1; q \rangle_{n-m}} \quad (177)$$

$$(-x)^m (-y)^n \text{QE} \left(\binom{n-m}{2} + n - m \right).$$

Solutions au voisinage de (∞, ∞) . Dès la seconde Ansatz nous obtenons les équations

$$\begin{cases} \{a_1 + \rho\}_q \{b_2 + \rho - \sigma\}_q = 0, \\ \{a_2 + \sigma\}_q \{b_1 + \sigma - \rho\}_q = 0. \end{cases} \quad (178)$$

Cela a les trois solutions

$$\begin{aligned} \rho &= -a_1, \quad \sigma = -a_2, \\ \rho &= -a_1, \quad \sigma = -a_1 - b_1, \\ \rho &= -a_2 - b_2, \quad \sigma = -a_2. \end{aligned} \quad (179)$$

Théorème 7.3. *La première solution est*

$$\begin{aligned} g_1(x, y; q) \\ = x^{-a_1} y^{-a_2} G_2(a_1; a_2; a_2 + b_2 - a_1, a_1 + b_1 - a_2 | q; x^{-1} q^{1-a_1}, y^{-1} q^{1-a_2}). \end{aligned} \quad (180)$$

Cette solution est un q -analogue de [12, p.32].

Démonstration. Pour le cas $\rho = -a_1$, $\sigma = -a_2$, De (102) on obtient la récurrence

$$\begin{cases} \frac{C_{m+1,n}}{C_{mn}} = \frac{\langle -a_1 - m, a_1 + b_1 - a_2 - n + m; q \rangle_1}{\langle -m - 1, a_2 + b_2 - a_1 - m - 1 + n; q \rangle_1} \\ \frac{C_{m,n+1}}{C_{mn}} = \frac{\langle -a_2 - n, a_2 + b_2 - a_1 - m + n; q \rangle_1}{\langle a_1 + b_1 - a_2 - n + m, -1 - n; q \rangle_1} \end{cases} \quad (181)$$

La solution de cette récurrence est

$$g_1(x, y; q) = x^{-a_1} y^{-a_2} \sum_{m,n=0}^{\infty} \frac{\langle a_1 + b_1 - a_2; q \rangle_{m-n} \langle a_2 + b_2 - a_1; q \rangle_{n-m}}{\langle -m; q \rangle_m \langle -n; q \rangle_n} \quad (182)$$

$$\langle -a_2 + 1 - n; q \rangle_n \langle -a_1 - m + 1; q \rangle_m x^{-m} y^{-n} \stackrel{\text{par [7, 6.14]}}{=} \text{LCD}.$$

□

Théorème 7.4. *La deuxième solution est*

$$g_2(x, y; q) = x^{-a_1} y^{-a_1 - b_1} \sum_{m,n=0}^{\infty} \frac{\langle b_1 + b_2; q \rangle_{n-m} \langle a_1 + b_1; q \rangle_n \langle a_1; q \rangle_m}{\langle 1; q \rangle_m \langle a_1 + b_1 + 1 - a_2; q \rangle_n \langle 1; q \rangle_{n-m}} \quad (183)$$

$$(-x)^m (-y)^n \text{QE} \left(-\binom{n-m}{2} + m(1 - a_1) - n(-2a_1 - 2b_1 + a_2 - 1) \right).$$

Cette solution est un q -analogue de [2, p.32].

Démonstration. De la récurrence (102)) on obtient

$$\begin{cases} \frac{C_{m+1,n}}{C_{mn}} = \frac{\langle -a_1 - m, b_2 + m - n; q \rangle_1}{\langle b_1 - m + n - 1, -m - 1; q \rangle_1} \\ \frac{C_{m,n+1}}{C_{mn}} = \frac{\langle -a_1 - b_1 - n, b_1 + b_2 - m + n; q \rangle_1}{\langle -n - 1 + m, a_2 - a_1 - b_1 - 1 - n; q \rangle_1} \end{cases} \quad (184)$$

Cela implique

$$g_2(x, y; q) = x^{-a_1} y^{-a_1 - b_1} \sum_{m,n=0}^{\infty} \langle b_1 + b_2; q \rangle_{n-m} x^m y^n \quad (185)$$

$$\frac{\langle -a_1 - b_1 - n + 1; q \rangle_n \langle -a_1 - m + 1; q \rangle_m}{\langle -m; q \rangle_m \langle a_2 - a_1 - b_1 - n; q \rangle_n \langle m - n; q \rangle_{n-m}} \stackrel{\text{par [7, 6.14]}}{=} \text{LCD}.$$

□

La solution g_3 s'ensuit par symétrie.

Solutions au voisinage de $(0, \infty)$. Dès le troisième Ansatz nous obtenons les équations

$$\begin{cases} \{\rho\}_q \{b_1 + \sigma - \rho\}_q = 0, \\ \{a_2 + \sigma\}_q \{b_1 + \sigma - \rho\}_q = 0. \end{cases} \quad (186)$$

Cela a les deux solutions

$$\begin{aligned} \rho &= 0, \quad \sigma = -a_2, \\ b_1 + \sigma - \rho &= 0. \end{aligned} \quad (187)$$

Cela donne la seule solution

$$\begin{aligned} g_1(x, y; q) &= y^{-a_2} \sum_{m, n=0}^{\infty} \frac{\langle a_2 + b_2; q \rangle_{m+n} \langle a_1; q \rangle_m \langle a_2; q \rangle_n}{\langle a_2 + 1 - b_1; q \rangle_{m+n} \langle 1; q \rangle_m \langle 1; q \rangle_n} \\ & \quad (-x)^m (-y)^{-n} \text{QE} \left(\binom{m+n}{2} + m(b_1 - a_2 - 1) + n(b_1 - 2a_2) \right). \end{aligned} \quad (188)$$

Cette solution est un q -analogue de [2, p.33].

Démonstration. De la récurrence (106) on obtient

$$\begin{cases} \frac{C_{m+1, n}}{C_{mn}} = \frac{\langle a_1 + m, a_2 + b_2 + m + n; q \rangle_1}{\langle 1 + m, b_1 - a_2 - m - n - 1; q \rangle_1}, \\ \frac{C_{m, n+1}}{C_{mn}} = \frac{\langle -n - a_2, a_2 + b_2 + m + n; q \rangle_1}{\langle -n - 1, b_1 - a_2 - m - n - 1; q \rangle_1}. \end{cases} \quad (189)$$

Nous trouvons

$$\begin{aligned} g_1(x, y; q) &= y^{-a_2} \sum_{m, n=0}^{\infty} \frac{\langle a_2 + b_2; q \rangle_{m+n} \langle a_1; q \rangle_m \langle 1 - a_2 - n; q \rangle_n}{\langle b_1 - a_2 - m - n; q \rangle_{m+n} \langle 1; q \rangle_m \langle -n; q \rangle_n} \\ & \quad x^m y^{-n} \text{ par } \stackrel{[7, 6.14]}{=} \text{LCD}. \end{aligned} \quad (190)$$

□

La solution du quatrième Ansatz suit par symétrie.

7.2. Le système de $H_{2,q}$. Solutions au voisinage de $(0, 0)$.

Par le premier Ansatz nous obtenons les équations

$$\begin{cases} \{\rho\}_q \{\rho + e - 1\}_q = 0, \\ \{\sigma\}_q \{\rho - \sigma + a\}_q = 0. \end{cases} \quad (191)$$

Cela a les quatre solutions

$$\begin{aligned} \rho &= 0, \quad \sigma = 0, \\ \rho &= 1 - e, \quad \sigma = 0, \\ \rho &= 0, \quad \sigma = a, \\ \rho &= 1 - e, \quad \sigma = a - e + 1. \end{aligned} \quad (192)$$

La première solution est $H_2(a; b; c; d; e|q; x, y)$.

Théorème 7.5. *La deuxième solution est*

$$h_2(x, y; q) = x^{1-e} H_2(a + 1 - e; b + 1 - e; c; d; 2 - e|q; x, y). \quad (193)$$

Cette solution est un q -analogue de [12, p.392].

Démonstration. Pour le cas $\rho = 1 - e$, $\sigma = 0$ on obtient de la récurrence (97)

$$\begin{cases} \frac{C_{m+1,n}}{C_{mn}} = \frac{\langle a + 1 - e + m - n, b + 1 - e + m; q \rangle_1}{\langle 1 + m, 2 - e + m; q \rangle_1}, \\ \frac{C_{m,n+1}}{C_{mn}} = \frac{\langle c + n, d + n; q \rangle_1}{\langle 1 + n, a - e + m - n; q \rangle_1}. \end{cases} \quad (194)$$

Nous trouvons

$$h_2(x, y; q) = x^{1-e} \sum_{m,n=0}^{\infty} \frac{\langle a + 1 - e; q \rangle_{m-n} \langle b + 1 - e; q \rangle_m \langle c, d; q \rangle_n}{\langle 1, 2 - e; q \rangle_m \langle 1; q \rangle_n} \quad (195)$$

$x^m y^n = \text{LCD}.$

□

Nous trouverons plus tard une expression q -intégrale pour cette fonction dans (244).

Théorème 7.6. *La troisième solution est*

$$h_3(x, y; q) = y^a \sum_{m,n=0}^{\infty} \frac{\langle b; q \rangle_m \langle a+c, a+d; q \rangle_n}{\langle 1, e; q \rangle_m \langle 1+a; q \rangle_n \langle 1; q \rangle_{n-m}} (-x)^m (-y)^n \text{QE} \left(-\binom{n-m}{2} + (n-m)^2 \right). \quad (196)$$

Cette solution est un q -analogue de [12, p.392].

Démonstration. Pour le cas $\rho = 0$, $\sigma = a$ on obtient de la récurrence (97)

$$\begin{cases} \frac{C_{m+1,n}}{C_{mn}} = \frac{\langle m-n, b+m; q \rangle_1}{\langle e+m, 1+m; q \rangle_1}, \\ \frac{C_{m,n+1}}{C_{mn}} = \frac{\langle a+c+n, a+d+n; q \rangle_1}{\langle a+n+1, m-1-n; q \rangle_1}. \end{cases} \quad (197)$$

Nous trouvons la solution

$$h_3(x, y; q) = y^a \sum_{m,n=0}^{\infty} \frac{\langle b; q \rangle_m \langle a+c, a+d; q \rangle_n}{\langle 1, e; q \rangle_m \langle 1+a; q \rangle_n \langle m-n; q \rangle_{n-m}} x^m y^n \stackrel{\text{par [7, 6.14]}}{=} \text{LCD}. \quad (198)$$

□

Théorème 7.7. *La quatrième solution est*

$$h_4(x, y; q) = x^{1-e} y^{a+1-e} \sum_{m,n=0}^{\infty} \frac{\langle b+1-e; q \rangle_m \langle c+a-e+1, d+a-e+1; q \rangle_n}{\langle 1, 2-e; q \rangle_m \langle 2+a-e; q \rangle_n \langle 1; q \rangle_{n-m}} (-x)^m (-y)^n \text{QE} \left(-\binom{n-m}{2} + (n-m)^2 \right). \quad (199)$$

Cette solution est un q -analogue de [12, p.392].

Démonstration. Pour le cas $\rho = 1-e$, $\sigma = a+1-e$ on obtient de la récurrence (97)

$$\begin{cases} \frac{C_{m+1,n}}{C_{mn}} = \frac{\langle m-n, b+1-e+m; q \rangle_1}{\langle 1+m, 2-e+m; q \rangle_1}, \\ \frac{C_{m,n+1}}{C_{mn}} = \frac{\langle c+a-e+1+n, d+a-e+1+n; q \rangle_1}{\langle 2+a-e+n, m-1-n; q \rangle_1}. \end{cases} \quad (200)$$

La solution de cette récurrence est

$$h_4(x, y; q) = x^{1-e} y^{a+1-e} \sum_{m,n=0}^{\infty} x^m y^{-n} \frac{\langle b+1-e; q \rangle_m \langle c+a-e+1, d+a-e+1; q \rangle_n}{\langle 1, 2-e; q \rangle_m \langle 2+a-e; q \rangle_n \langle m-n; q \rangle_{n-m}} \text{ par [7, 6.14] LCD.} \quad (201)$$

□

Solutions au voisinage de (∞, ∞) .

Dès la seconde Ansatz nous obtenons les équations

$$\begin{cases} \{a + \rho - \sigma\}_q \{b + \rho\}_q = 0, \\ \{c + \sigma\}_q \{d + \sigma\}_q = 0. \end{cases} \quad (202)$$

Cela a les quatre solutions

$$\begin{aligned} \rho &= -b, \quad \sigma = -c, \\ \rho &= -b, \quad \sigma = -d, \\ \rho &= -a - d, \quad \sigma = -d, \\ \rho &= -a - c, \quad \sigma = -c. \end{aligned} \quad (203)$$

Théorème 7.8. *La première solution est*

$$h_1(x, y; q) = x^{-b} y^{-c} \sum_{m,n=0}^{\infty} \frac{\langle a+c-b; q \rangle_{n-m} \langle b-e+1, b; q \rangle_m \langle c; q \rangle_n}{\langle 1; q \rangle_m \langle 1, c-d+1; q \rangle_n} (-x)^{-m} (-y)^{-n} \text{QE} \left(\binom{n}{2} - \binom{m}{2} + m(e-2b) + n(2-d) \right). \quad (204)$$

Cette solution est un q -analogue de [2, p.33].

Démonstration. Pour le cas $\rho = -b$, $\sigma = -c$ on obtient de la récurrence (102)

$$\begin{cases} \frac{C_{m+1,n}}{C_{mn}} = \frac{\langle e-b-m-1, -b-m; q \rangle_1}{\langle a+c-b-m-1+n, -1-m; q \rangle_1}, \\ \frac{C_{m,n+1}}{C_{mn}} = \frac{\langle -c-n, a+c-b-m+n; q \rangle_1}{\langle d-c-n-1, -1-n; q \rangle_1}. \end{cases} \quad (205)$$

nous trouvons

$$h_1(x, y; q) = x^{-b}y^{-c} \sum_{m,n=0}^{\infty} x^{-m}y^{-n} \langle a+c-b; q \rangle_{n-m} \frac{\langle e-b-m, -b-m+1; q \rangle_m \langle -c+1-n; q \rangle_n}{\langle -m; q \rangle_m \langle -n, d-c-n; q \rangle_n} \text{ par [7, 6.14] LCD.} \quad (206)$$

□

La deuxième solution s'ensuit par symétrie dans les deux paramètres c et d .

Théorème 7.9. *La troisième solution est*

$$h_3(x, y; q) = x^{-a-d}y^{-d} \sum_{m,n=0}^{\infty} \frac{\langle a+d, a+d+1-e; q \rangle_m \langle d; q \rangle_n}{\langle a+d+1-b; q \rangle_m \langle 1, d+1-c; q \rangle_n \langle 1; q \rangle_{m-n}} x^{-m}y^{-n} \text{QE} \left(n^2 - mn + m(1+e-a-d-b) + n(1-c) \right). \quad (207)$$

Cette solution est un q -analogue de [2, p.33].

Démonstration. Pour le cas $\rho = -a-d$, $\sigma = -d$ on obtient de la récurrence (102)

$$\begin{cases} \frac{C_{m+1,n}}{C_{mn}} = \frac{\langle e-a-d-m-1, -a-d-m; q \rangle_1}{\langle n-m-1, b-a-d-m-1; q \rangle_1}, \\ \frac{C_{m,n+1}}{C_{mn}} = \frac{\langle -m+n, -d-n; q \rangle_1}{\langle c-d-n-1, -1-n; q \rangle_1}. \end{cases} \quad (208)$$

La solution de cette récurrence est

$$h_3(x, y; q) = x^{-a-d}y^{-d} \sum_{m,n=0}^{\infty} x^{-m}y^{-n} \frac{\langle e-a-d-m, -a-d-m+1; q \rangle_m \langle -d-n+1; q \rangle_n}{\langle b-a-d-m; q \rangle_m \langle c-d-n, -n; q \rangle_n \langle n-m, m-n; q \rangle_n} \text{ par [7, 6.14] LCD.} \quad (209)$$

□

La quatrième solution suit par symétrie dans c et d .

Solutions au voisinage de $(0, \infty)$. Dès le troisième Ansatz nous obtenons les équations

$$\begin{cases} \{\rho\}_q \{e + \rho - 1\}_q = 0, \\ \{c + \sigma\}_q \{d + \sigma\}_q = 0. \end{cases} \quad (210)$$

Cela a les quatre solutions

$$\begin{aligned} \rho = 0, \quad \sigma = -c, \\ \rho = 0, \quad \sigma = -d, \\ \rho = 1 - e, \quad \sigma = -c, \\ \rho = 1 - e, \quad \sigma = -d. \end{aligned} \quad (211)$$

Théorème 7.10. *La première solution est*

$$\begin{aligned} h_1(x, y; q) &= y^{-c} \sum_{m,n=0}^{\infty} \frac{\langle a + c; q \rangle_{m+n} \langle b; q \rangle_m \langle c; q \rangle_n}{\langle 1, e; q \rangle_m \langle 1, c + 1 - d; q \rangle_n} x^m (-y)^{-n} \\ &\text{QE} \left(\frac{n^2}{2} + n \left(-d + \frac{3}{2} \right) \right). \end{aligned} \quad (212)$$

Cette solution est un q -analogue de [2, p.34].

Démonstration. De la récurrence (106) on obtient

$$\begin{cases} \frac{C_{m+1,n}}{C_{mn}} = \frac{\langle a + c + m + n, b + m; q \rangle_1}{\langle e + m, 1 + m; q \rangle_1}, \\ \frac{C_{m,n+1}}{C_{mn}} = \frac{\langle -c - n, a + c + m + n; q \rangle_1}{\langle d - c - n - 1, -1 - n; q \rangle_1}. \end{cases} \quad (213)$$

nous trouvons

$$\begin{aligned} h_1(x, y; q) &= y^{-c} \sum_{m,n=0}^{\infty} \frac{\langle a + c; q \rangle_{m+n} \langle b; q \rangle_m \langle -c - n + 1; q \rangle_n}{\langle 1, e; q \rangle_m \langle -n, d - c - n; q \rangle_n} \\ &x^m y^{-n} \text{ par } [7, 6.14] \text{ LCD.} \end{aligned} \quad (214)$$

□

La deuxième solution suit par symétrie dans c et d .

Théorème 7.11. *La troisième solution est*

$$h_3(x, y; q) = x^{1-e} y^{-c} \sum_{m,n=0}^{\infty} \frac{\langle a+c+1-e; q \rangle_{m+n} \langle b+1-e; q \rangle_m}{\langle 1, 2-e; q \rangle_m \langle 1, c+1-d; q \rangle_n} \langle c; q \rangle_n x^m (-y)^{-n} \text{QE} \left(\frac{n^2}{2} + n(-d + \frac{3}{2}) \right). \quad (215)$$

Cette solution est un q -analogue de [2, p.34].

Démonstration. De la récurrence (106) on obtient

$$\begin{cases} \frac{C_{m+1,n}}{C_{mn}} = \frac{\langle a+c+1-e+m+n, b+1-e+m; q \rangle_1}{\langle 2-e+m, 1+m; q \rangle_1}, \\ \frac{C_{m,n+1}}{C_{mn}} = \frac{\langle -c-n, a+c+1-e+m+n; q \rangle_1}{\langle d-c-n-1, -1-n; q \rangle_1}. \end{cases} \quad (216)$$

nous trouvons

$$h_1(x, y; q) = y^{-c} \sum_{m,n=0}^{\infty} \frac{\langle a+c+1-e; q \rangle_{m+n} \langle b+1-e; q \rangle_m \langle -c-n+1; q \rangle_n}{\langle 1, 2-e; q \rangle_m \langle -n, d-c-n; q \rangle_n} x^m y^{-n} \stackrel{\text{par [7, 6.14]}}{=} \text{LCD}. \quad (217)$$

□

La quatrième solution suit par symétrie dans c et d .

Solutions au voisinage de $(\infty, 0)$. A partir du quatrième Ansatz nous obtenons les équations

$$\begin{cases} \{a+\rho-\sigma\}_q \{b+\rho\}_q = 0, \\ \{\sigma\}_q \{a+\rho-\sigma\}_q = 0. \end{cases} \quad (218)$$

Cela a la seule solution

$$\rho = -b, \sigma = 0. \quad (219)$$

Théorème 7.12. *La solution est*

$$h_1(x, y; q) = x^{-b} \sum_{m,n=0}^{\infty} \frac{\langle b+1-e, b; q \rangle_m \langle c, d; q \rangle_n}{\langle b+1-a; q \rangle_{m+n} \langle 1; q \rangle_m \langle 1; q \rangle_n} x^{-m} (-y)^n$$

$$\text{QE} \left(\frac{n^2}{2} + mn + m(e-a-b+1) + n \left(b-a + \frac{1}{2} \right) \right).$$
(220)

Cette solution est un q -analogue de [2, p.34].

Démonstration. De la récurrence (110) on obtient

$$\begin{cases} \frac{C_{m+1,n}}{C_{mn}} = \frac{\langle e-b-m-1, -b-m; q \rangle_1}{\langle a-b-m-n-1, -m-1; q \rangle_1}, \\ \frac{C_{m,n+1}}{C_{mn}} = \frac{\langle c+n, d+n; q \rangle_1}{\langle a-b-m-n-1, 1+n; q \rangle_1}. \end{cases}$$
(221)

Nous trouvons

$$h_1(x, y; q) = x^{-b} \sum_{m,n=0}^{\infty} \frac{\langle e-b-m, -b-m+1; q \rangle_m \langle c, d; q \rangle_n}{\langle -m; q \rangle_m \langle 1; q \rangle_n \langle a-b-m-n; q \rangle_{m+n}}$$

$$x^m y^{-n} \text{ par } \stackrel{[7, 6.14]}{=} \text{LCD.}$$
(222)

□

8. LES FONCTIONS DE q -HORN "ANORMALES"

Afin de définir les fonction "anormales" de q -Horn, nous avons besoin d'une notation pour simplifier les formules.

Définition 20. On définit la forme quadratique suivante dans les entiers positifs m et n .

$$\mathcal{Q}(m, n, a) \equiv -\binom{n}{2} + n(m+a-1),$$
(223)

La motivation pour les prochaines définitions est que les formules d'opérateur dans [3] nécessitent des facteurs q supplémentaires. Notez que le premier indice rappelle la fonction de Horn correspondante, et le deuxième indice dénote des variantes avec différents facteurs q .

Définition 21. Les fonctions anormales de q -Horn sont définis par

$$G_{11}(a; b; b' | q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1+m_2} \langle b; q \rangle_{m_2-m_1} \langle b'; q \rangle_{m_1-m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} \quad (224)$$

$$QE(\mathcal{Q}(0, m_2, b') + \mathcal{Q}(m_2, m_1, b)) x_1^{m_1} x_2^{m_2}.$$

$$G_{21}(a; a'; b, b' | q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1} \langle a'; q \rangle_{m_2} \langle b; q \rangle_{m_2-m_1} \langle b'; q \rangle_{m_1-m_2}}{\langle 1; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} \quad (225)$$

$$QE(\mathcal{Q}(0, m_2, b') + \mathcal{Q}(m_2, m_1, b)) x_1^{m_1} x_2^{m_2}.$$

$$H_{11}(a; b; c; d | q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1-m_2} \langle b; q \rangle_{m_1+m_2} \langle c; q \rangle_{m_2}}{\langle 1, d; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} \quad (226)$$

$$QE(\mathcal{Q}(0, m_2, a)) x_1^{m_1} x_2^{m_2}.$$

$$H_{12}(a; b; c; d | q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1-m_2} \langle b; q \rangle_{m_1+m_2} \langle c; q \rangle_{m_2}}{\langle 1, d; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} \quad (227)$$

$$QE(\mathcal{Q}(m_1, m_2, a)) x_1^{m_1} x_2^{m_2}.$$

$$H_{21}(a; b; c; d; e | q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1-m_2} \langle b; q \rangle_{m_1} \langle c, d; q \rangle_{m_2}}{\langle 1, e; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} \quad (228)$$

$$QE(\mathcal{Q}(m_1, m_2, a)) x_1^{m_1} x_2^{m_2}.$$

$$H_{22}(a; b; c; d; e | q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a; q \rangle_{m_1-m_2} \langle b; q \rangle_{m_1} \langle c, d; q \rangle_{m_2}}{\langle 1, e; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} \quad (229)$$

$$QE(\mathcal{Q}(0, m_2, a)) x_1^{m_1} x_2^{m_2}.$$

$$H_{23}(a+1-e; b+1-e; c; d; 2-e | q; x_1, x_2) \equiv \sum_{m_1, m_2=0}^{\infty} \frac{\langle a+1-e; q \rangle_{m_1-m_2} \langle b+1-e; q \rangle_{m_1} \langle c, d; q \rangle_{m_2}}{\langle 1, 2-e; q \rangle_{m_1} \langle 1; q \rangle_{m_2}} \quad (230)$$

$$QE(\mathcal{Q}(0, m_2, a-e+1)) x_1^{m_1} x_2^{m_2}.$$

Pour le théorème suivant, notez que nous faisons un double changement de signe dans les arguments pour les deux premières fonctions. Ensuite, nous effectuons un seul changement de signe dans les arguments. Nous avons corrigé certaines des formules dans [3].

Théorème 8.1. *Un q -analogue des formules d'opérateur [3, p. 810].*

$$\begin{aligned} & {}_2\phi_1(a_1, b_2 - \theta_{q,2}; 1 - b_1 - \theta_{q,2}|q; x_1) {}_2\phi_1(a_2, b_1; 1 - b_2|q; x_2) \\ & = G_{21}(a_1, a_2; b_1, b_2|q; -x_1, -x_2). \end{aligned} \quad (231)$$

Un q -analogue de la version corrigée de [3, p. 819].

$$\begin{aligned} & {}_2\phi_1(a - \theta_{q,2}, b + \theta_{q,2}; d|q; x_1) {}_2\phi_1(b, c; 1 - a|q; x_2) \\ & = H_{11}(a; b; c; d|q; x_1, -x_2). \end{aligned} \quad (232)$$

$$\begin{aligned} & {}_2\phi_1(b + \theta_{q,1}, c; 1 - a - \theta_{q,1}|q; x_2) {}_2\phi_1(a, b; d|q; x_1) \\ & = H_{12}(a; b; c; d|q; x_1, -x_2). \end{aligned} \quad (233)$$

$$\begin{aligned} & {}_2\phi_1(c, d; 1 - a - \theta_{q,1}|q; x_2) {}_2\phi_1(a, b; e|q; x_1) \\ & = H_{21}(a; b; c; d; e|q; x_1, -x_2). \end{aligned} \quad (234)$$

$$\begin{aligned} & {}_2\phi_1(b, a - \theta_{q,2}; e|q; x_1) {}_2\phi_1(c, d; 1 - a|q; x_2) \\ & = H_{22}(a; b; c; d; e|q; x_1, -x_2). \end{aligned} \quad (235)$$

La formule suivante donne une expression d'opérateur pour la deuxième solution à l'anormal q -analogue du système canonique pour $H_2(a; b; c; d; e; x_1, -x_2)$. Un q -analogique de [3, p. 820]. Comparez avec Horn [12, p. 392].

$$\begin{aligned} & x_1^{1-e} {}_2\phi_1(a + 1 - e - \theta_{q,2}, b + 1 - e; 2 - e|q; x_1) {}_2\phi_1(c, d; e - a|q; x_2) \\ & = x_1^{1-e} H_{23}(a + 1 - e; b + 1 - e; c; d; 2 - e|q; x_1, -x_2). \end{aligned} \quad (236)$$

Démonstration. On utilise la formule [7, 6.15]. □

Théorème 8.2. Une représentation q -intégrale de $G_{11}(a; b_1; b_2|q; -x, -y)$.
Un q -analogue de la version corrigée de [3, p. 809].

$$\begin{aligned} G_{11}(a; b_1; b_2|q; -x, -y) &\cong \Gamma_q \left[\begin{matrix} 1 - b_1 \\ b_2, 1 - b_1 - b_2 \end{matrix} \right] \int_0^1 t^{b_2-1} \frac{(qt; q)_{-b_1-b_2}}{(xt; q)_a} \\ &\sum_{m=0}^{\infty} \frac{\langle a; q \rangle_m}{\langle 1; q \rangle_m (xtq^a; q)_m} \left(\frac{y}{t} \right)^m \text{QE}(m(b_1 + b_2 - 1)) d_q(t). \end{aligned} \quad (237)$$

Démonstration.

$$\begin{aligned} \text{LCG} \stackrel{\text{par [7, 7.50], (19)}}{=} \Gamma_q \left[\begin{matrix} 1 - b_1 - \theta_{q,2} \\ b_2 - \theta_{q,2}, 1 - b_1 - b_2 \end{matrix} \right] \int_0^1 t^{b_2-\theta_{q,2}-1} \frac{(qt; q)_{-b_1-b_2}}{(xt; q)_{a+\theta_{q,2}}} \\ {}_2\phi_1(a, b; 1 - b_2|q; y) d_q(t) \stackrel{\text{par (77)}}{=} \text{LCD}. \end{aligned} \quad (238)$$

□

Théorème 8.3. Une représentation q -intégrale de $G_{21}(a_1, a_2; b_1, b_2|q; -x, -y)$.
Un q -analogue de [3, p. 810].

$$\begin{aligned} G_{21}(a_1, a_2; b_1, b_2|q; -x, -y) &\cong \Gamma_q \left[\begin{matrix} 1 - b_1 \\ b_2, 1 - b_1 - b_2 \end{matrix} \right] \int_0^1 t^{b_2-1} \frac{(qt; q)_{-b_1-b_2}}{(xt; q)_{a_1}} \\ &\frac{1}{\left(\frac{y}{t} q^{b_1+b_2-1}; q \right)_{a_2}} d_q(t). \end{aligned} \quad (239)$$

Démonstration. Nous utilisons le théorème q -binomial.

$$\begin{aligned} \text{LCG} \stackrel{\text{par [7, 7.50]}}{=} \Gamma_q \left[\begin{matrix} 1 - b_1 - \theta_{q,2} \\ b_2 - \theta_{q,2}, 1 - b_1 - b_2 \end{matrix} \right] \int_0^1 t^{b_2-1} \frac{(qt; q)_{-b_1-b_2}}{(xt; q)_{a_1}} \\ {}_2\phi_1(a_2, b_1; 1 - b_2|q; y) d_q(t) \stackrel{\text{par (77)}}{=} \stackrel{[7, 7.27]}{=} \text{LCD}. \end{aligned} \quad (240)$$

□

Théorème 8.4. Une représentation q -intégrale de $H_{11}(a; b; c; d|q; x, -y)$.
Un q -analogue de la version corrigée de [3, p. 819].

$$\begin{aligned} H_{11}(a; b; c; d|q; x, -y) &\cong \Gamma_q \left[\begin{matrix} d \\ b, d-b \end{matrix} \right] \int_0^1 t^{b-1} \frac{(qt; q)_{d-b-1}}{(xt; q)_a} \\ &\sum_{m=0}^{\infty} \frac{\langle c, 1-d+b; q \rangle_m (tq^{d-b}; q)_{-m}}{\langle 1, 1-a; q \rangle_m (xtq^a; q)_{-m}} (-ty)^m \\ &\text{QE} \left(-\binom{m}{2} + m(d-b-1) \right) d_q(t). \end{aligned} \quad (241)$$

Démonstration.

$$\begin{aligned} \text{LCG} \stackrel{\text{par [7, 7.50]}}{=} \Gamma_q \left[\begin{matrix} d \\ b + \theta_{q,2}, d-b-\theta_{q,2} \end{matrix} \right] \int_0^1 t^{b+\theta_{q,2}-1} \frac{(qt; q)_{d-b-\theta_{q,2}-1}}{(xt; q)_{a-\theta_{q,2}}} \\ {}_2\phi_1(b, c; 1-a|q; y) d_q(t) \stackrel{\text{par (77)}}{=} \text{LCD}. \end{aligned} \quad (242)$$

□

Théorème 8.5. Une représentation q -intégrale de $H_{22}(a; b; c; d; e|q; x, -y)$.
Un q -analogue de [3, p. 820], [5, 3.4].

$$\begin{aligned} H_{22}(a; b; c; d; e|q; x, -y) &\cong \Gamma_q \left[\begin{matrix} e \\ b, e-b \end{matrix} \right] \int_0^1 t^{b-1} \frac{(qt; q)_{e-b-1}}{(xt; q)_a} \\ &\sum_{m=0}^{\infty} \frac{\langle c, d; q \rangle_m}{\langle 1, 1-a; q \rangle_m (xtq^a; q)_{-m}} y^m d_q(t). \end{aligned} \quad (243)$$

Théorème 8.6. Une représentation q -intégrale de

$$H_{23}(a+1-e; b+1-e; c; d; 2-e|q; x, -y).$$

Un q -analogue de [3, p. 820].

$$\begin{aligned} &H_{23}(a+1-e; b+1-e; c; d; 2-e|q; x, -y) \\ &\cong \Gamma_q \left[\begin{matrix} 2-e \\ b+1-e, 1-b \end{matrix} \right] \int_0^1 t^{b-e} \frac{(qt; q)_{-b}}{(xt; q)_{a+1-e}} \\ &\sum_{m=0}^{\infty} \frac{\langle c, d; q \rangle_m}{\langle 1, e-a; q \rangle_m (xtq^{a+1-e}; q)_{-m}} y^m d_q(t). \end{aligned} \quad (244)$$

Démonstration. On utilisons (236).

□

9. LES ÉQUATIONS AUX q -DIFFÉRENCES POUR LES FONCTIONS
(NORMALE) CONFLUENTES DE q -HORN

Les systèmes suivantes d'équations q -différence pour les fonctions q -Horn normales confluentes sont simplement prouvés.

(1) Système $\Gamma_1(a_1; b_1, b_2|q; x, y)$

$$\begin{cases} \{-\theta_{q,1} + \theta_{q,2} + b_1 - 1\}_q D_{q,x} - \{\theta_{q,1} + a_1\}_q \{\theta_{q,1} - \theta_{q,2} + b_2\}_q = 0, \\ \{\theta_{q,1} - \theta_{q,2} + b_2 - 1\}_q (1 - q) D_{q,y} - \{-\theta_{q,1} + \theta_{q,2} + b_1\}_q = 0. \end{cases} \quad (245)$$

(2) Système $\Gamma_2(b_1, b_2|q; x, y)$

$$\begin{cases} \{-\theta_{q,1} + \theta_{q,2} + b_1 - 1\}_q (1 - q) D_{q,x} - \{\theta_{q,1} - \theta_{q,2} + b_2\}_q = 0, \\ \{\theta_{q,1} - \theta_{q,2} + b_2 - 1\}_q (1 - q) D_{q,y} - \{-\theta_{q,1} + \theta_{q,2} + b_1\}_q = 0. \end{cases} \quad (246)$$

(3) Système $\mathcal{H}_1(a; b; d|q; |q; x, y)$

$$\begin{cases} \{\theta_{q,1} + d\}_q D_{q,x} - \{\theta_{q,1} - \theta_{q,2} + a\}_q \{\theta_{q,1} + \theta_{q,2} + b\}_q = 0, \\ \{\theta_{q,1} - \theta_{q,2} + a - 1\}_q (1 - q) D_{q,y} - \{\theta_{q,1} + \theta_{q,2} + b\}_q = 0. \end{cases} \quad (247)$$

(4) Système $\mathcal{H}_2(a; b; d; e|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + e\}_q D_{q,x} - \{\theta_{q,1} - \theta_{q,2} + a\}_q \{\theta_{q,1} + b\}_q = 0, \\ \{\theta_{q,1} - \theta_{q,2} + a - 1\}_q (1 - q) D_{q,y} - \{\theta_{q,2} + d\}_q = 0. \end{cases} \quad (248)$$

(5) Système $\mathcal{H}_3(a; b; e|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + e\}_q D_{q,x} - \{\theta_{q,1} - \theta_{q,2} + a\}_q \{\theta_{q,1} + b\}_q = 0, \\ \{\theta_{q,1} - \theta_{q,2} + a - 1\}_q (1 - q)^2 D_{q,y} - \mathbf{I} = 0. \end{cases} \quad (249)$$

(6) Système $\mathcal{H}_4(a; d; e|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + d\}_q (1 - q) D_{q,x} - \{\theta_{q,1} - \theta_{q,2} + a\}_q = 0, \\ \{\theta_{q,1} - \theta_{q,2} + a - 1\}_q (1 - q) D_{q,y} - \{\theta_{q,2} + c\}_q = 0. \end{cases} \quad (250)$$

(7) Système $\mathcal{H}_5(a; d|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + d\}_q (1 - q) D_{q,x} - \{\theta_{q,1} - \theta_{q,2} + a\}_q = 0, \\ \{\theta_{q,1} - \theta_{q,2} + a - 1\}_q (1 - q)^2 D_{q,y} - \mathbf{I} = 0. \end{cases} \quad (251)$$

(8) Système $\mathcal{H}_6(a; c|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + \theta_{q,2} + c\}_q D_{q,x} - \{2\theta_{q,1} + \theta_{q,2} + a\}_q \{2\theta_{q,1} + \theta_{q,2} + a + 1\}_q = 0, \\ \{\theta_{q,1} + \theta_{q,2} + c\}_q (1 - q) D_{q,y} - \{2\theta_{q,1} + \theta_{q,2} + a\}_q = 0. \end{cases} \quad (252)$$

(9) Système $\mathcal{H}_7(a; c; d|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + c\}_q D_{q,x} - \{2\theta_{q,1} + \theta_{q,2} + a\}_q \{2\theta_{q,1} + \theta_{q,2} + a + 1\}_q = 0, \\ \{\theta_{q,2} + d\}_q (1 - q) D_{q,y} - \{2\theta_{q,1} + \theta_{q,2} + a\}_q = 0. \end{cases} \quad (253)$$

(10) Système $\mathcal{H}_8(a; b|q; x, y)$

$$\begin{cases} \{-\theta_{q,1} + \theta_{q,2} + b - 1\}_q D_{q,x} - \{2\theta_{q,1} - \theta_{q,2} + a\}_q \{2\theta_{q,1} - \theta_{q,2} + a + 1\}_q = 0, \\ \{2\theta_{q,1} - \theta_{q,2} + a - 1\}_q (1 - q) D_{q,y} - \{-\theta_{q,1} + \theta_{q,2} + b\}_q = 0. \end{cases} \quad (254)$$

(11) Système $\mathcal{H}_9(a; c, d|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + d\}_q D_{q,x} - \{2\theta_{q,1} - \theta_{q,2} + a\}_q \{2\theta_{q,1} - \theta_{q,2} + a + 1\}_q = 0, \\ \{2\theta_{q,1} - \theta_{q,2} + a - 1\}_q (1 - q) D_{q,y} - \{\theta_{q,2} + c\}_q = 0. \end{cases} \quad (255)$$

(12) Système $\mathcal{H}_{10}(a; c, d|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + d\}_q D_{q,x} - \{2\theta_{q,1} - \theta_{q,2} + a\}_q \{2\theta_{q,1} - \theta_{q,2} + a + 1\}_q = 0, \\ \{2\theta_{q,1} - \theta_{q,2} + a - 1\}_q (1 - q)^2 D_{q,y} - I = 0. \end{cases} \quad (256)$$

(13) Système $\mathcal{H}_{11}(a; c; d; e|q; x, y)$

$$\begin{cases} \{\theta_{q,1} + e\}_q (1 - q) D_{q,x} - \{\theta_{q,1} - \theta_{q,2} + a\}_q = 0, \\ \{\theta_{q,1} - \theta_{q,2} + a - 1\}_q D_{q,y} - \{\theta_{q,2} + c\}_q \{\theta_{q,2} + d\}_q = 0. \end{cases} \quad (257)$$

10. LES q -INTÉGRALES D'EULER DES FONCTIONS 'ANORMAL' CONFLUENTES DE q -HORN

Pour prouver ces intégrales de q -Euler, nous prenons les limites des fonctions anormales du q -Horn correspondantes et utilisons le théorème q -binomial.

Théorème 10.1. Une représentation q -intégrale de $\Gamma_{11}(a; b_1, b_2|q; -x, -y)$.

$$\Gamma_{11}(a; b_1, b_2|q; -x, -y) \cong \Gamma_q \left[\begin{matrix} 1 - b_1 \\ b_2, 1 - b_1 - b_2 \end{matrix} \right] \int_0^1 t^{b_2-1} \frac{(qt; q)_{-b_1-b_2}}{(xt; q)_a} d_q(t). \quad (258)$$

Théorème 10.2. Une représentation q -intégrale de $\Gamma_{21}(b_1, b_2|q; -x, -y)$.

$$\Gamma_{21}(b_1, b_2|q; -x, -y) \cong \Gamma_q \left[\begin{matrix} 1 - b_1 \\ b_2, 1 - b_1 - b_2 \end{matrix} \right] \int_0^1 t^{b_2-1} (qt; q)_{-b_1-b_2} d_q(t). \quad (259)$$

Théorème 10.3. Une représentation q -intégrale de $\mathcal{H}_{11}(a; b; d|q; x, -y)$.

$$\begin{aligned} \mathcal{H}_{11}(a; b; d|q; x, -y) &\cong \Gamma_q \left[\begin{matrix} d \\ b, d - b \end{matrix} \right] \int_0^1 t^{b-1} \frac{(qt; q)_{d-b-1}}{(xt; q)_a} \\ &\sum_{m=0}^{\infty} \frac{\langle 1 - d + b; q \rangle_m (tq^{d-b} q)_{-m}}{\langle 1, 1 - a; q \rangle_m (xtq^a, q)_{-m}} (-ty)^m \\ &\text{QE} \left(-\binom{m}{2} + m(d - b - 1) \right) d_q(t). \end{aligned} \quad (260)$$

Théorème 10.4. Une représentation q -intégrale de $\mathcal{H}_{22}(a; b; d; e|q; x, -y)$.

$$\begin{aligned} \mathcal{H}_{22}(a; b; d; e|q; x, -y) &\cong \Gamma_q \left[\begin{matrix} e \\ b, e - b \end{matrix} \right] \int_0^1 t^{b-1} \frac{(qt; q)_{e-b-1}}{(xt; q)_a} \\ &\sum_{m=0}^{\infty} \frac{\langle d; q \rangle_m}{\langle 1, 1 - a; q \rangle_m (xtq^a; q)_{-m}} y^m d_q(t). \end{aligned} \quad (261)$$

Théorème 10.5. Une représentation q -intégrale.

$$\begin{aligned} \mathcal{H}_{32}(a; b; e|q; x, -y) &\cong \Gamma_q \left[\begin{matrix} e \\ b, e - b \end{matrix} \right] \int_0^1 t^{b-1} \frac{(qt; q)_{e-b-1}}{(xt; q)_a} \\ &\sum_{m=0}^{\infty} \frac{1}{\langle 1, 1 - a; q \rangle_m (xtq^a; q)_{-m}} y^m d_q(t). \end{aligned} \quad (262)$$

Théorème 10.6. *Une représentation q -intégrale.*

$$\begin{aligned} & \mathcal{H}_{23}(a+1-e; b+1-e; d; 2-e|q; x, -y) \\ & \cong \Gamma_q \left[\begin{matrix} 2-e \\ b+1-e, 1-b \end{matrix} \right] \int_0^1 t^{b-e} \frac{(qt; q)_{-b}}{(xt; q)_{a+1-e}} \\ & \sum_{m=0}^{\infty} \frac{\langle d; q \rangle_m}{\langle 1, e-a; q \rangle_m} \frac{y^m}{(xtq^{a+1-e}; q)_{-m}} d_q(t). \end{aligned} \quad (263)$$

Théorème 10.7. *Une représentation q -intégrale de*

$$\begin{aligned} & \mathcal{H}_{33}(a+1-e; b+1-e; 2-e|q; x, -y). \\ & \mathcal{H}_{33}(a+1-e; b+1-e; 2-e|q; x, -y) \\ & \cong \Gamma_q \left[\begin{matrix} 2-e \\ b+1-e, 1-b \end{matrix} \right] \int_0^1 t^{b-e} \frac{(qt; q)_{-b}}{(xt; q)_{a+1-e}} \\ & \sum_{m=0}^{\infty} \frac{1}{\langle 1, e-a; q \rangle_m} \frac{y^m}{(xtq^{a+1-e}; q)_{-m}} d_{\bullet}(t). \end{aligned} \quad (264)$$

11. CONCLUSION

Nous avons trouvé les systèmes canoniques d'équations aux q -différences pour tous les fonctions de q -Appell confluentes, pour les dix fonctions normales de q -Horn, leurs formes confluentes, et ainsi de suite. Il faut remarquer que les systèmes canoniques des équations différentielles des dix fonctions de Horn en cette forme n'existe pas dans la littérature sauf en autres formes dans [3]. Puis, nous avons examiné les autres solutions des systèmes des équations aux q -différences en trois formes

- (1) la forme factorisée umbrale
- (2) le développement en série, avec des régions de convergence, q -analogues de [3]
- (3) éventuellement, une représentation q -intégrale.

Grace au caractère spécial de ces formules, nous pouvons employer le théorème q -binomial pour aller aux limites et de trouver des formules simples pour toutes les fonctions confluentes. Ces formules q -intégrales sont intéressants, car elles donnent des prolongements méromorphes aux définitions sérielles respectives.

Plutôt, ces formules n'étaient pas connues auparavant sous forme hypergéométrique, donc cet article est d'une grande valeur pour les chercheurs s'occupants des fonctions hypergéométriques multiple.

Nous n'avons pas rendu toutes les fonctions (normale) confluyente de q -Horn aux celles qui sont les fonctions (anormale), puisque les q -intégrales correspondantes demanderont trop de place.

12. DISCUSSION

Nous remercions Amédée Debiard et Bernard Gaveau pour des articles très intéressants sur les fonctions hypergéométrique multiples. Nous avons gardé leur notation autant que possible.

Nous observons que nos systèmes d'équations opérationnelles de q -différences pour les fonctions multiples sont totalement différentes des systèmes des équations aux dérivées partielles dans [6, p. 233 et suiv.]. Il n'existe aucune preuve de ces équations aux dérivées partielles dans la littérature. Nos systèmes sont plus simples, parce que c'est facile à vérifier si une puissance donnée ou une série de Laurent satisfait au système.

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Notre but prochain objectif est de traiter les problèmes correspondants pour les fonctions de q -Lauricella et des fonctions pareilles d'un ordre plus haut.

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MULTISET CONGRUENCES ON GROUPS

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Abstract

In this paper, we define a multiset congruence on a group and investigate their fundamental properties. In particular, we show that there exists a one-to-one correspondence between the set of normal multigroups over a group and the set of multiset congruences on this group.

Keywords: Multiset; Multiset relation; Normal multigroup; Multiset congruence.

1 Introduction

In classical set theory, a set is a well-defined collection of distinct objects. If the restriction of distinctness on the nature of the objects forming a set is relaxed, then a mathematical structure known as multiset is obtained. Thus, a multiset differs from a set in the sense that each element has a multiplicity and a natural number not necessarily one that indicates how many times it is a member of the multiset. One of the most natural and simplest examples is the multiset of prime factors of a positive integer n . The number 200 has $2^3 5^2$ which gives the number $[2, 5]_{3,2}$. Also, the cubic equation $x^3 - 5x^2 + 3x + 9 = 0$ has roots 3, 3, and -1 which give the multiset $[3, -1]_{2,1}$.

Classical set theory is a basic concept to represent various situations in mathematical notation where repeated occurrences of elements are not allowed. However, in various circumstances, repetition of elements become mandatory to the system. For example, in a graph with loops, repeated hydrogen atoms in water molecule, repeated observations in statistical samples and so on. With the development of multiset theory, numerous contributions have been done by utilizing the concept of multisets from simple theoretical to logical and innovative disciplines. The theoretical aspect of multiset theory serves as a tool which extends the classical structure of algebra into the new form of mathematical structures such as multiset relations [1], multiset topologies [2], multiset proximity [4], multiset filters [3] and n -level multisets [7]. Keeping this in mind, Nazmul *et al.* [5] made an excellent contribution in generalizing groups via multiset theory. The authors considered the initial universal set X to be a group through which a multiset is drawn. Since then, many researchers have followed their definition for further investigations (see, for example, [6],[9],[8],[10] and [11]).

The recent definition of multigroup shows a strong analogy in the behaviour of group and makes it possible to extend some of the major notions and results of groups to that of multigroups. This motivated us to extend the relationships that exist between normal subgroups and congruences in classical sense to the case of multigroup theory. In this paper, we define a multiset congruence on a group and investigate their fundamental properties.

2 Preliminaries

In this section, we give some basic definitions needed for this paper from [1] and [5].

Definition 2.1 *An unordered collection of objects with repetition allowed is a multiset. Formally if X is a set of elements, a multiset A drawn from X is characterized by a count function C_A defined as $C_A : X \rightarrow N_{\geq 0} = \{0, 1, 2, \dots\}$.*

For each $x \in X$, $C_A(x)$ denotes the number of occurrences of the element x in A . The representation of the multiset A drawn from $X = \{x_1, x_2, \dots, x_n\}$ is $\{x_1/m_1, x_2/m_2, \dots, x_n/m_n\}$ such that x_i appears m_i times in A , ($i = 1, 2, \dots, n$).

The usual set operations can be carried to multisets. Let $A_1, A_2 \in X$. Then

- (i) $A_1 \subseteq A_2$ if $C_{A_1}(x) \leq C_{A_2}(x)$, $\forall x \in X$.
- (ii) Let $A = A_1 \cap A_2$ such that $\forall x \in X$, $C_A(x) = C_{A_1} \wedge C_{A_2}(x)$.
- (iii) Let $A = A_1 \cup A_2$ such that $\forall x \in X$, $C_A(x) = C_{A_1} \vee C_{A_2}(x)$.

Definition 2.2 *Let X and Y be two nonempty sets and $f : X \rightarrow Y$ be a mapping. Then*

- (i) *the image $f(A)$ of an mset $A \in [X]^n$ is defined as*

$$C_{f(A)}(y) = \begin{cases} \bigvee_{f(x)=y} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & f^{-1}(y) = \emptyset \end{cases} .$$
- (ii) *the inverse image $f^{-1}(B)$ of an mset $B \in [Y]^n$ is defined as $C_{f^{-1}B}(x) = C_B(f(x))$.*

It is easy to see that $f(A)$ and $f^{-1}(B)$ are multisets.

Definition 2.3 *Let A and B be two multisets drawn from a set X . The Cartesian product of A and B is defined as $A \times B = \{(m/x, n/y) / mn : x \in^m A, y \in^n B\}$.*

Definition 2.4 *A multiset relation $\mathfrak{R}$ from a multiset A to B , symbolized as $\mathfrak{R} : A \longrightarrow B$ is any submultiset of $A \times B$ where every element of $\mathfrak{R}$ has a count. Also, m/x is $\mathfrak{R}$ -related to n/y is symbolized as $(m/x, n/y) \in \mathfrak{R}$. Formally, $\mathfrak{R} = \{(m/x, n/y) / mn : (m/x, n/y) \in^{mn} \mathfrak{R}\}$.*

An entry of the form $(m/x, n/y) \in^{mn} \mathfrak{R}$ means that the pair (x, y) occurs in mn times in $\mathfrak{R}$.

3 Multigroups

This section briefly describes the concept of multigroups and outlines some basic results needed for this paper except Proposition 3.3.

Definition 3.1 [5] *Let X be any group. By a multigroup over X we mean a count function $C_A : X \longrightarrow N_{\geq 0}$ such that*

$$C_A(ab^{-1}) \geq C_A(a) \wedge C_A(b), \quad \forall a, b \in X$$

Moreover, a normal multigroup over X is defined as a multigroup satisfying the condition

$$C_A(ab) \geq C_A(ba)$$

We denote by $MG(X)$ ($NMG(X)$) the set of all multigroups (normal multigroups) over X .

Throughout the paper, let e be the identity element of X .

Proposition 3.1 [5] *Let $A \in MG(X)$. Then*

- (i) $C_A(e) \geq C_A(a)$,
- (ii) $C_A(a) = C_A(a^{-1})$.

Proposition 3.2 [6] *Let $A \in MG(X)$. Then $C_A(ab^{-1}) = C_A(e)$ implies $C_A(a) = C_A(b)$.*

The next result is an extension of classical homomorphism.

Proposition 3.3 *Let $f : X \rightarrow Y$ be a homomorphism and $A \in NMG(X)$. Then $X/f^{-1}(A) \cong f(X)/A$.*

proof

Define $f : X/f^{-1}(A) \rightarrow f(X)/A$ by $f(a/f^{-1}(A)) = f(a)/A$ for all $a \in X$.

If $a/f^{-1}(A) = b/f^{-1}(A)$ and since $C_{f^{-1}(A)}(ab^{-1}) = C_{f^{-1}(A)}(e)$ by definition, then we have $C_A(f(a)(f(b))^{-1}) = C_A(f(e)) = C_A(e')$, where $e' \in Y$. Thus, $f(a)/A = f(b)/A$. This means that f is well-defined.

Suppose that $f(a/f^{-1}(A)) = f(b/f^{-1}(A))$ that is, $f(a)/A = f(b)/A$. It follows from definition that $C_A(f(a)(f(b))^{-1}) = C_A(e')$ and hence $C_{f^{-1}(A)}(ab^{-1}) = C_{f^{-1}(A)}(e)$ implies $a/f^{-1}(A) = b/f^{-1}(A)$. Therefore, f is injective.

It is easy to prove that f is a surjective homomorphism. Thus, $X/f^{-1}(A) \cong f(X)/A$.

4 Main results

Definition 4.1 *Let X be a group. A binary function $C_{\mathfrak{R}} : X \times X \rightarrow N_{\geq 0}$ is called a multiset congruence on X if the following conditions are satisfied for all $a, b, c, d \in X$:*

- (i) $C_{\mathfrak{R}}(e, e) \geq C_{\mathfrak{R}}(a, a)$,
- (ii) $C_{\mathfrak{R}}(a, b) = C_{\mathfrak{R}}(b, a)$,
- (iii) $C_{\mathfrak{R}}(a, c) \geq C_{\mathfrak{R}}(a, b) \wedge C_{\mathfrak{R}}(b, c)$,
- (iv) $C_{\mathfrak{R}}(a, b) \wedge C_{\mathfrak{R}}(c, d) \leq C_{\mathfrak{R}}(ac, bd)$.

The set of all multiset congruences on X is denoted by $MC(X)$.

Proposition 4.1 *If $\mathfrak{R} \in MC(X)$ and $a, b, c \in X$, then*

- (i) $C_{\mathfrak{R}}(a, b) \geq C_{\mathfrak{R}}(ac, bc) \wedge C_{\mathfrak{R}}(ca, cb)$,
- (ii) $C_{\mathfrak{R}}(a^{-1}, b^{-1}) = C_{\mathfrak{R}}(a, b)$.

proof

(i) The second part of this proof is similar to the first part, so we show only the first part.

$$\begin{aligned} (a)C_{\mathfrak{R}}(a, b) &= C_{\mathfrak{R}}(acc^{-1}, bcc^{-1}) \\ &\geq C_{\mathfrak{R}}(ac, bc) \end{aligned} \quad (1)$$

Similarly,

$$C_{\mathfrak{R}}(a, b) \geq C_{\mathfrak{R}}(ca, cb) \quad (2)$$

Therefore, combining (1) and (2) we obtain (i).

(ii)

$$\begin{aligned} C_{\mathfrak{R}}(a^{-1}, b^{-1}) &\leq C_{\mathfrak{R}}(aa^{-1}, ab^{-1}) \\ &= C_{\mathfrak{R}}(e, ab^{-1}) \\ &\leq C_{\mathfrak{R}}(eb, ab^{-1}b) \\ &= C_{\mathfrak{R}}(b, a) = C_{\mathfrak{R}}(a, b) \end{aligned}$$

Conversely,

$$\begin{aligned} C_{\mathfrak{R}}(a, b) &\leq C_{\mathfrak{R}}(a^{-1}a, a^{-1}b) \\ &= C_{\mathfrak{R}}(e, a^{-1}b) \\ &\leq C_{\mathfrak{R}}(eb^{-1}, a^{-1}bb^{-1}) \\ &= C_{\mathfrak{R}}(b^{-1}, a^{-1}) = C_{\mathfrak{R}}(a^{-1}, b^{-1}) \end{aligned}$$

Hence, $C_{\mathfrak{R}}(a^{-1}, b^{-1}) = C_{\mathfrak{R}}(a, b)$.

Proposition 4.2 *If $\mathfrak{R} \in MC(X)$, then $C_{\mathfrak{R}}(a, b) = C_{\mathfrak{R}}(ab^{-1}, e)$, $\forall a, b \in X$.*

proof

$$C_{\mathfrak{R}}(a, b) \leq C_{\mathfrak{R}}(ab^{-1}, bb^{-1})$$

$$\begin{aligned}
 &= C_{\mathfrak{R}}(ab^{-1}, e) \\
 &\leq C_{\mathfrak{R}}(ab^{-1}b, eb) \\
 &= C_{\mathfrak{R}}(a, b)
 \end{aligned}$$

Hence, $C_{\mathfrak{R}}(a, b) = C_{\mathfrak{R}}(ab^{-1}, e)$.

Proposition 4.3 *If $\mathfrak{R} \in MC(X)$, then $X/\mathfrak{R}$ forms a group.*

Proof

The proof is straightforward.

Proposition 4.4 *If $A \in NMG(X)$, then the multiset relation $\mathfrak{R}_A(a, b)$ defined by $C_{\mathfrak{R}_A}(a, b) = C_A(ab^{-1})$ is a multiset congruence.*

proof

Conditions (i) and (ii) are obvious from Definition 4.1. We only show (iii) and (iv).

For the case of (iii), we have

$$\begin{aligned}
 C_{\mathfrak{R}_A}(a, c) &= C_A(ac^{-1}) \\
 &= C_A(ab^{-1}bc^{-1}) \\
 &\geq C_A(ab^{-1}) \wedge C_A(bc^{-1}) \\
 &= C_{\mathfrak{R}_A}(a, b) \wedge C_{\mathfrak{R}_A}(b, c)
 \end{aligned}$$

For the case of (iv), it follows that

$$\begin{aligned}
 C_{\mathfrak{R}_A}(a, b) \wedge C_{\mathfrak{R}_A}(c, d) &= C_A(ab^{-1}) \wedge C_A(cd^{-1}) \\
 &= C_A(b^{-1}a) \wedge C_A(cd^{-1}) \\
 &\leq C_A(b^{-1}(acd^{-1})) \\
 &= C_A(acd^{-1}b^{-1}) \\
 &= C_A((ac)(bd)^{-1}) \\
 &= C_{\mathfrak{R}_A}(ac, bd)
 \end{aligned}$$

Hence, $\mathfrak{R}_A \in MC(X)$.

Proposition 4.5 *If $\mathfrak{R} \in MC(X)$, then the function $C_{A_{\mathfrak{R}}}$ from X to $N_{\geq 0}$ defined by $C_{A_{\mathfrak{R}}}(a) = C_{\mathfrak{R}}(a, e)$ is a normal multigroup over X .*

proof

Clearly,

$$\begin{aligned} C_{A_{\mathfrak{R}}}(a) \wedge C_{A_{\mathfrak{R}}}(b) &= C_{\mathfrak{R}}(a, e) \wedge C_{\mathfrak{R}}(b, e) \\ &\leq C_{\mathfrak{R}}(ab^{-1}, e) = C_{A_{\mathfrak{R}}}(ab^{-1}) \end{aligned}$$

Thus $C_{A_{\mathfrak{R}}}$ is a multigroup of X .

Moreover,

$$\begin{aligned} C_{A_{\mathfrak{R}}}(ab) = C_{\mathfrak{R}}(ab, e) &\leq C_{\mathfrak{R}}(abb^{-1}, eb^{-1}) = C_{\mathfrak{R}}(a, b^{-1}) \\ &\leq C_{\mathfrak{R}}(a^{-1}a, a^{-1}b^{-1}) = C_{\mathfrak{R}}(e, a^{-1}b^{-1}) \\ &\leq C_{\mathfrak{R}}(eb, a^{-1}b^{-1}b) = C_{\mathfrak{R}}(b, a^{-1}) \\ &\leq C_{\mathfrak{R}}(ba, a^{-1}a) = C_{\mathfrak{R}}(ba, e) = C_{A_{\mathfrak{R}}}(ba) \end{aligned}$$

Hence, $C_{A_{\mathfrak{R}}}$ is a normal multigroup over X .

Proposition 4.6 *Let $\mathfrak{R}_1, \mathfrak{R}_2 \in MC(X)$. Then $A_{\mathfrak{R}_1 \cap \mathfrak{R}_2} = A_{\mathfrak{R}_1} \cap A_{\mathfrak{R}_2}$.*

proof

Let $a \in X$ and by Proposition 4.5,

$$\begin{aligned} C_{A_{\mathfrak{R}_1 \cap \mathfrak{R}_2}}(a) &= C_{\mathfrak{R}_1 \cap \mathfrak{R}_2}(a, e) \\ &= C_{\mathfrak{R}_1}(a, e) \wedge C_{\mathfrak{R}_2}(a, e) \\ &= C_{A_{\mathfrak{R}_1}}(a) \wedge C_{A_{\mathfrak{R}_2}}(a) = C_{A_{\mathfrak{R}_1} \cap A_{\mathfrak{R}_2}}(a) \end{aligned}$$

Hence, $A_{\mathfrak{R}_1 \cap \mathfrak{R}_2} = A_{\mathfrak{R}_1} \cap A_{\mathfrak{R}_2}$.

Proposition 4.7 (i) *If $A \in NMG(X)$, then $A = A_{\mathfrak{R}}$.*

(ii) *If $\mathfrak{R} \in MC(X)$, then $\mathfrak{R} = \mathfrak{R}_A$.*

Proof

- (i) The proof is straightforward.
- (ii) Let $a, b \in X$. Then

$$C_{\mathfrak{R}_A}(a, b) = C_A(ab^{-1}) = C_{\mathfrak{R}}(ab^{-1}, e) \geq C_{\mathfrak{R}}(a, b)$$

Conversely,

$$\begin{aligned} C_{\mathfrak{R}}(a, b) \geq C_{\mathfrak{R}}(ab^{-1}, e) &= C_{\mathfrak{R}}(e, ab^{-1}) \\ &= C_A(ab^{-1}) = C_{\mathfrak{R}_A}(a, b) \end{aligned}$$

Hence, $\mathfrak{R} = \mathfrak{R}_A$.

Proposition 4.8 *There exists a bijective $f : NMG(X) \longrightarrow MC(X)$.*

proof

Define a mapping $f : NMG(X) \longrightarrow MC(X)$ by $f(A) = C_{\mathfrak{R}_A}(a, b)$.

If $A = B$, then for any $a, b \in X$, we have $C_{\mathfrak{R}_A}(a, b) = C_A(ab^{-1}) = C_B(ab^{-1}) = C_{\mathfrak{R}_B}(a, b)$.

Thus, $f(A) = C_{\mathfrak{R}_A}(a, b) = C_{\mathfrak{R}_B}(a, b) = f(B)$. This implies that f is well-defined.

If $f(A) = f(B)$ and $A, B \in NMG(X)$, then for every $a \in X$, $C_A(a) = C_A(ae^{-1}) = C_{\mathfrak{R}_A}(a, e) = C_{\mathfrak{R}_B}(a, e) = C_B(ae^{-1}) = C_B(ae) = C_B(a)$ and so $A = B$. Hence, f is one-one.

Let $\wp \in MC(X)$. Then by Propositions 4.4 and 4.5 the result follows. That is, for all

$x, y \in X$, $C_{\mathfrak{R}_{A_{\wp}}}(a, b) = C_{A_{\wp}}(ab^{-1}) = C_{\wp}(ab^{-1}, e) = C_{\wp}(a, b)$. Thus, $f(A_{\wp}) = C_{\mathfrak{R}_{A_{\wp}}}(a, b) = C_{\wp}(a, b)$.

Hence, f is onto. Therefore, we have that φ is a bijection from $NMG(X)$ to $MC(X)$.

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BELL, BERNOULLI, CAUCHY, HARMONIC AND STIRLING NUMBERS

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Abstract

We show the companion relation of an identity deduced by Akiyama-Tanigawa for the Stirling numbers, and we exhibit that it implies an expression involving Bell and Stirling numbers. Besides, we give an elementary proof of a property obtained by Boyadzhiev for Cauchy numbers. Finally, we indicate how the Bernoulli numbers can be generated via Cauchy and Stirling numbers of the second kind or via a determinant in terms of Stirling numbers of the first kind.

Keywords: Stirling, Harmonic, Cauchy, Bernoulli, and Bell numbers; Determinants.

1.- Introduction

Akiyama-Tanigawa [1-5] obtained the identity:

$$\sum_{r=k}^n r S_n^{[r]} S_r^{(k)} = (-1)^{n+k} \binom{n}{k-1}, \quad 0 \leq k \leq n, \quad (1)$$

where $S_m^{(j)}$ and $S_q^{[l]}$ are the Stirling numbers of the first and second kind, respectively [5-7].

We deduced, via arithmetical experimentation, the property [8-10]:

$$\sum_{r=k}^n r S_n^{(r)} S_r^{[k]} = (-1)^{n+k+1} (n-k-1)! \binom{n}{k-1}, \quad 1 \leq 1+k \leq n, \quad (2)$$

In Sec. 2 we exhibit several relations implied by (1) and (2). The Sec. 3 has an elementary deduction of an identity of Sun for Cauchy numbers. In Sec. 4 we indicate several forms to write the Bernoulli numbers in terms of a determinant involving Stirling numbers of the first kind, in particular, we correct a formula of Qi-Chapman.

2. Identities involving Stirling, Harmonic, Bell, Bernoulli, and Cauchy numbers

We have the following values [6, 11] in terms of the harmonic numbers [12-14]:

$$\begin{aligned} S_r^{(1)} &= (-1)^{r+1} (r-1)!, \quad S_r^{(2)} = (-1)^r (r-1)! H_{r-1}, \quad H_m = \sum_{j=1}^m \frac{1}{j}, \\ S_r^{(3)} &= \frac{1}{2} (-1)^{r+1} (r-1)! (H_{r-1}^2 - H_{r-1}^{(2)}), \quad H_k^{(2)} = \sum_{q=1}^k \frac{1}{q^2}, \end{aligned} \quad (3)$$

whose application into (1) for $k = 1, 2, 3$ gives the expressions:

$$\begin{aligned}
 \sum_{r=1}^n (-1)^r r! S_n^{[r]} &= (-1)^n, \quad n \geq 1; \\
 \sum_{r=2}^n (-1)^r r! S_n^{[r]} H_{r-1} &= (-1)^n n, \quad n \geq 2, \\
 \sum_{r=3}^n (-1)^r r! S_n^{[r]} (H_{r-1}^2 - H_{r-1}^{(2)}) &= (-1)^n n(n-1), \quad n \geq 3.
 \end{aligned} \tag{4}$$

Similarly, the identity (2) with $k = 1, 2, 3$ and the following values for $r \geq 1$ [6, 15]:

$$S_r^{[1]} = 1, \quad S_r^{[2]} = 2^{r-1} - 1, \quad S_r^{[3]} = \frac{1}{2} (3^{r-1} + 1 - 2^r), \tag{5}$$

imply the relations:

$$\begin{aligned}
 \sum_{r=1}^n r S_n^{(r)} &= \begin{cases} 1, & n = 1, \\ (-1)^n (n-2)!, & n \geq 2, \end{cases} \\
 \sum_{r=2}^n r 2^r S_n^{(r)} &= \begin{cases} 8, & n = 2, \\ 2(-1)^n (n-3)! n(n-3), & n \geq 3, \end{cases} \\
 \sum_{r=3}^n r 3^r S_n^{(r)} &= \begin{cases} 81, & n = 3, \\ 3(-1)^n (n-4)! \left[\frac{n(n-1)}{(n-3)(n-4) - 6(n-1)(n-2)H_{n-1}} \right], & n \geq 4. \end{cases}
 \end{aligned} \tag{6}$$

Thus, it is evident that (1) and (2) provide several identities for various values of k , involving the Stirling and harmonic numbers. The application of $\sum_{k=0}^n$ to (2) gives the interesting property:

$$\frac{1}{n} \sum_{k=1}^n k S_n^{(k)} B(k) = 1 + (n-1)! \sum_{j=0}^{n-2} \frac{(-1)^{n+j}}{j! (n-j) (n-j-1)}, \quad n \geq 1, \tag{7}$$

involving the Bell numbers [6, 16, 17].

We know the identity [6]:

$$\frac{x^n}{(1-x)(1-2x)\cdots(1-nx)} = \sum_{r=0}^{\infty} S_r^{[n]} x^r, \quad (8)$$

and the Cauchy numbers [11]:

$$c_n \equiv \sum_{k=0}^n \frac{1}{k+1} S_n^{(k)} = \int_0^1 [t]_n dt, \quad (9)$$

where $[x]_n$ is the descending factorial function [18, 19], then from (9) and (8) with $x = -1$:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+x)(1+2x)\cdots(1+nx)} c_n x^{n+1} &= \\ \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} (-1)^r x^{r+1} \sum_{k=0}^n \frac{1}{k+1} S_n^{(k)} S_r^{[n]}, &= \\ \sum_{r=0}^{\infty} (-1)^r x^{r+1} \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{n=k}^{\infty} S_n^{(k)} S_r^{[n]} &= \\ \sum_{r=0}^{\infty} (-1)^r x^{r+1} \sum_{k=0}^{\infty} \frac{1}{k+1} \delta_{kr}, & \\ = \sum_{r=0}^{\infty} \frac{(-1)^r}{r+1} x^{r+1} = \ln(1+x), & \quad |x| < 1, \end{aligned} \quad (10)$$

relation obtained by Boyadzhiev [20].

In [21, 22] we have the identity:

$$\sum_{r=0}^n \frac{1}{r+1} B_{r+1} S_n^{(r)} = -\frac{1}{n+1} c_{n+1}, \quad (11)$$

involving Bernoulli numbers [5, 6], whose inversion [23] gives the relation:

$$B_{n+1} = -(n+1) \sum_{r=0}^n \frac{1}{r+1} c_{r+1} S_n^{[r]}, \quad n \geq 0, \quad (12)$$

that is, the Cauchy and Stirling numbers of the second kind permit generate the Bernoulli numbers.

3. Sun's identity involving Cauchy numbers

Here we consider the quantities:

$$Q_n \equiv \sum_{k=0}^n \frac{a_{n-k}}{k+1}, \quad n \geq 1, \quad (13)$$

for the sequence [24, 25]:

$$a_n = (-1)^n \int_0^1 \binom{x}{n} dx = \frac{(-1)^n}{n!} c_n, \quad (14)$$

that is:

$$a_0 = 1, \quad a_1 = -\frac{1}{2}, \quad a_2 = -\frac{1}{12}, \quad a_3 = -\frac{1}{24}, \quad a_4 = -\frac{19}{720},$$

$$a_5 = -\frac{3}{160}, \dots \quad (15)$$

From (13) for several values of n :

$$Q_1 = a_1 + \frac{1}{2} a_0, \quad Q_2 + a_1 Q_1 = 0, \quad Q_3 + a_1 Q_2 + a_2 Q_1 = 0, \quad (16)$$

$$Q_4 + a_1 Q_3 + a_2 Q_2 + a_3 Q_1 = 0,$$

$$Q_5 + a_1 Q_4 + a_2 Q_3 + a_3 Q_2 + a_4 Q_1 = 0, \dots$$

but (15) implies $Q_1 = 0$, then (16) gives $Q_2 = Q_3 = \dots = 0$, hence $Q_n = 0$, $n \geq 1$, which is the identity obtained by Zhi-Wei Sun [24]:

$$\sum_{k=0}^n \frac{a_{n-k}}{k+1} = 0, \quad n \geq 1. \quad (17)$$

The system (16) means $\sum_{k=0}^n a_{n-k} Q_k = 0$, where we can apply (13) and (14) to deduce:

$$a_n = \sum_{r=0}^n \frac{(-1)^{n-r}}{r+1} \int_0^1 dx \int_0^1 dy \binom{x+y}{n-r}, \quad (18)$$

because [26]:

$$\sum_{k=r}^n \binom{x}{n-k} \binom{y}{k-r} = \binom{x+y}{n-r}. \quad (19)$$

It is possible to show the property:

$$\int_0^1 dy \sum_{k=0}^n \frac{(-1)^k}{k+1} \binom{x+y}{n-k} = \binom{x}{n}, \quad n \geq 0, \quad (20)$$

then (18) reproduces (14). For $x = 0$, $n \geq 1$, the identity (20) gives (17); with $x = -1$ the relation (20) implies:

$$\sum_{k=0}^n \frac{b_{n-k}}{k+1} = 1, \quad b_n \equiv (-1)^n \int_{-1}^0 \binom{x}{n} dx, \quad n \geq 0. \quad (21)$$

The expression (20) can be written in the form:

$$\binom{x}{n} = \frac{(-1)^n}{n+1} + \int_x^{x+1} d\eta \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} \binom{\eta}{n-k}, \quad n \geq 1, \quad (22)$$

then it is immediate to obtain the known results [5, 6]:

$$\begin{aligned} \left[\frac{d}{dx} \binom{x}{n} \right]_{x=-1} &= (-1)^{n+1} H_n, & \left[\frac{d}{dx} \binom{x}{n} \right]_{x=0} &= \frac{(-1)^{n+1}}{n}, \\ \left[\frac{d}{dx} \binom{x}{n} \right]_{x=n} &= H_n, \end{aligned} \quad (23)$$

$$\left[\frac{d}{dx} \binom{x}{n} \right]_{x=1} = \begin{cases} 1, & n = 1, \\ \frac{(-1)^n}{n(n-1)}, & n \geq 2, \end{cases}$$

involving harmonic numbers [27].

The binomial coefficients have relationship with Stirling numbers of the first kind [8, 28]:

$$\binom{x}{k} = \frac{1}{k!} \sum_{m=0}^k S_k^{(m)} x^m, \quad (24)$$

then (20) gives the following identity:

$$S_n^{(m)} = \frac{1}{n+1} \sum_{j=m}^n \binom{j}{m} \frac{1}{j+1-m} \sum_{q=0}^n (-1)^q q! \binom{n+1}{n-q} S_{n-q}^{(j)},$$

$$n \geq m \geq 0. \quad (25)$$

4. Bernoulli numbers in terms of a determinant

Here we show several forms to write the Bernoulli numbers in terms of a determinant, in particular, we correct a formula of Qi-Chapman. In [29] we find the following interesting determinant:

$$Det(x, \lambda_j) \equiv \begin{vmatrix} \lambda_0 & 1 & & & & \\ \lambda_1 & S_2^{(1)}x & 1 & & & \\ \lambda_2 & S_3^{(1)}x^2 & S_3^{(2)}x & 1 & & \\ \dots & \dots & \dots & \dots & \dots & \\ \lambda_{n-2} & S_{n-1}^{(1)}x^{n-2} & \dots & & 1 & \\ \lambda_{n-1} & S_n^{(1)}x^{n-1} & \dots & \dots & \dots & S_n^{(n-1)}x \end{vmatrix} =$$

$$(-1)^{n-1} \sum_{k=0}^n \lambda_{k-1} x^{n-k} S_n^{[k]}, \quad n \geq 1, \quad (26)$$

involving the Stirling numbers. If we select special values for x and λ_j then it is possible to generate several types of numbers in terms of a determinant, for example [30]:

If $x = -1$, $\lambda_k = \frac{(k+1)!}{k+2}$, then from (26):

$$B_n \equiv \sum_{k=0}^n \frac{(-1)^k k!}{k+1} S_n^{[k]} = -Det\left(-1, \frac{(j+1)!}{j+2}\right), \quad (27)$$

hence:

$$B_2 = - \begin{vmatrix} 1/2 & 1 \\ 2/3 & 1 \end{vmatrix} = \frac{1}{6}, \quad B_3 = - \begin{vmatrix} 1/2 & 1 & 0 \\ 2/3 & 1 & 1 \\ 3/2 & 2 & 3 \end{vmatrix} = 0,$$

$$B_4 = - \begin{vmatrix} 1/2 & 1 & 0 & 0 \\ 2/3 & 1 & 1 & 0 \\ 3/2 & 2 & 3 & 1 \\ 24/5 & 6 & 11 & 6 \end{vmatrix} = -\frac{1}{30}, \dots \quad (28)$$

The expressions (26)-(28) show that the Bernoulli numbers defined in terms of Stirling numbers of the second kind can be obtained via a determinant involving Stirling numbers of the first kind.

There are several forms to write B_n in terms of a determinant:

a). Turnbull [31] and Booth-Nguyen [32].

$$B_n = \frac{(-1)^{n-1}}{(n+1)!} \begin{vmatrix} 1 & 2 & 0 & 0 & \dots & 0 \\ 1 & 3 & 3 & 0 & \dots & 0 \\ 1 & 4 & 6 & 4 & \dots & 0 \\ 1 & 5 & 10 & 10 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ (n+1) & (n+1) & (n+1) & (n+1) & \dots & (n+1) \\ 0 & 1 & 2 & 3 & \dots & (n-1) \end{vmatrix}. \quad (29)$$

b). Costabile-Dell'Accio-Gualtieri [33].

$$B_n = \frac{(-1)^n}{(n-1)!} \begin{vmatrix} 1/2 & 1/3 & 1/4 & \dots & 1/n & \frac{1}{n} + 1 \\ 1 & 1 & 1 & \dots & 1 & 1 \\ & & 3 & \dots & n-1 & n \\ 0 & 2 & \binom{3}{2} & \dots & \binom{n-1}{2} & \binom{n}{2} \\ 0 & 0 & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \binom{n-1}{n-2} & \binom{n}{n-2} \end{vmatrix}. \quad (30)$$

c). Qi – Chapman [34].

$$B_n = (-1)^n \text{Det} (B_{lm}), \quad 1 \leq l \leq n, \quad 0 \leq m \leq n-1, \quad (31)$$

where $B_{lm} = \frac{1}{l+1} \binom{l+1}{m}$, which is incorrect, then our correction is:

$$B_{lm} = \begin{cases} 0, & m = l+1, \\ \frac{1}{l+1} \binom{l+1}{m}, & m \neq l+1, \end{cases} \quad (32)$$

thus:

$$B_2 = \begin{vmatrix} 1/2 & 1 \\ 1/3 & 1 \end{vmatrix} = \frac{1}{6}, \quad B_3 = - \begin{vmatrix} 1/2 & 1 & 0 \\ 1/3 & 1 & 1 \\ 1/4 & 1 & 3/2 \end{vmatrix} = 0, \\ B_4 = \begin{vmatrix} 1/2 & 1 & 0 & 0 \\ 1/3 & 1 & 1 & 0 \\ 1/4 & 1 & 3/2 & 1 \\ 1/5 & 1 & 2 & 2 \end{vmatrix} = -\frac{1}{30}, \dots \quad (33)$$

in agreement with (28).

d). Woon [35], Chen [36] and Larson [37].

$$B_n = \begin{vmatrix} \frac{1}{2!} & 1 & 0 & \dots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & \dots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \dots & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \dots & \frac{1}{2!} \end{vmatrix}. \quad (34)$$

Therefore the determinants (27), (29), (30), (32) and (34) are equal to zero for $n = 3, 5, 7, 9, \dots$. In [30] it was applied (26) for different values of x and λ_k to obtain special types of numbers, for example, if $x = -1$, $\lambda_k = (-1)^k$, then from (26):

$$\text{Bell numbers} = B(n) \equiv \sum_{k=0}^n S_n^{[k]} = \text{Det}(-1, (-1)^j). \quad (35)$$

In (27) we see that the Bernoulli numbers are in terms of Stirling numbers of the second kind, which also can be written as a determinant [38]:

$$S_n^{[k]} = \frac{1}{1! 2! \dots k!} \begin{vmatrix} 1 & \dots & 1^{k-1} & 1^n \\ 2 & \dots & 2^{k-1} & 2^n \\ \vdots & \vdots & \vdots & \vdots \\ k & \dots & k^{k-1} & k^n \end{vmatrix}, \quad (36)$$

that is:

$$S_4^{[2]} = \frac{1}{2} \begin{vmatrix} 1 & 1 \\ 2 & 16 \end{vmatrix} = 7, \quad S_5^{[3]} = \frac{1}{12} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 32 \\ 3 & 9 & 243 \end{vmatrix} = 25, \\ S_n^{[1]} = |1| = 1, \quad n \geq 1, \dots \quad (37)$$

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HELIX-HYPEROPERATIONS ON LIE-SANTILLI ADMISSIBILITY

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Abstract

The Lie-Santilli Admissibility can be transferred in hyperstructure theory. Hyperstructure theory can overcome restrictions which ordinary algebraic structures have. A hyperproduct on non-square ordinary matrices can be defined by using the so called helix-hyperoperations. We present the hyper-Lie-Santilli's admissibility in the general case and focus on the helix-hyperoperations. We present applications using special types of ordinary non-square matrices.

Keywords: hyperoperation, helix-hyperoperation, H_v -structures, H_v -fields, Lie-Santilli's admissibility.

1. INTRODUCTION

The H_V -structures, the largest class of hyperstructures, satisfy the weak axioms where the non-empty intersection replaces the equality. Some basic definitions and results [1], [2], [5], [18], [19], [20]:

Algebraic hyperstructure is a set H equipped with at least one *hyperoperation* $\cdot: H \times H \rightarrow P(H) - \{\emptyset\}$. The *weak associativity* is $(xy)z \cap x(yz) \neq \emptyset$, $\forall x, y, z \in H$ and *weak commutativity*: $xy \cap yx \neq \emptyset$, $\forall x, y \in H$. $(H, \cdot)$ is *H_V -semigroup* if it is weak associative and it is *H_V -group* if, moreover, it is reproductive, i.e. $xH = Hx = H$, $\forall x \in H$.

Motivation. The quotient of a group by invariant subgroup, is a group. F. Marty (1934): the quotient of a group by a subgroup is a hypergroup. T. Vougiouklis (1990): the quotient of a group by a partition is an H_V -group.

$(R, +, \cdot)$ is an *H_V -ring* if $(+)$ and $(\cdot)$ are weak associative, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \emptyset, \quad (x+y)z \cap (xz+yz) \neq \emptyset, \quad \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ H_V -ring, $(M, +)$ a weak commutative H_V -group and there is a hyperoperation

$$\cdot : R \times M \rightarrow P(M): (a, x) \rightarrow ax$$

such that, $\forall a, b \in R, \forall x, y \in M$

$$a(x+y) \cap (ax+ay) \neq \emptyset, \quad (a+b)x \cap (ax+bx) \neq \emptyset, \quad (ab)x \cap a(bx) \neq \emptyset,$$

then M is an *H_V -module* over F . In the case of an H_V -field F , which is defined later, instead of an H_V -ring R , then the *H_V -vector space* is defined.

For more definitions and applications on H_V -structures see books and papers as [1], [2], [5], [10], [19], [20], [21], [22].

Let $(H, \cdot)$, $(H, *)$ be H_V -semigroups on the same H , the hyperoperation $(\cdot)$ is *smaller* than $(*)$, and $(*)$ *greater* than $(\cdot)$, if and only if, there exists an

$$f \in \text{Aut}(H, *) \text{ for which } xy \subset f(x*y), \quad \forall x, y \in H.$$

The Little Theorem. Greater hyperoperations than ones which are weak associative or weak commutative, are weak associative or weak commutative, respectively.

An H_V -structure is *minimal* if it contains no other corresponding H_V -structure.

The *fundamental relations* β^* , γ^* and ε^* , are defined, in H_v -groups, H_v -rings and H_v -vector spaces, respectively, as the smallest equivalences so that the quotient is group, ring and vector space, respectively.

Theorem 1.1 Let $(H, \cdot)$ be an H_v -group and denote U the set of all finite products of elements of H . Define β in H by setting $x\beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then β^* is the transitive closure of β .

An element is called *single* if its fundamental class is singleton.

More general structures can be defined by using the fundamental structures.

Definition 1.2 An H_v -ring $(R, +, \cdot)$ is called *H_v -field* if R/γ^* is a field.

Definition 1.3 [5], [12], [32] Let $(L, +)$ be an H_v -vector space over the H_v -field $(F, +, \cdot)$, $\varphi: F \rightarrow F/\gamma^*$, the canonical map and $\omega_F = \{x \in F: \varphi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ^* . Let ω_L be the core of the canonical map $\varphi': L \rightarrow L/\varepsilon^*$ and denote again 0 the zero of L/ε^* . Consider the *bracket hyperoperation*:

$$[,]: L \times L \rightarrow P(L): (x, y) \rightarrow [x, y]$$

then L is an *H_v -Lie algebra* over F if the following axioms are satisfied:

(L1) The bracket hyperoperation is bilinear, i.e.

$$\begin{aligned} & [\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \emptyset \\ & [x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \emptyset, \quad \forall x, x_1, x_2, y, y_1, y_2 \in L \text{ and } \lambda_1, \lambda_2 \in F \end{aligned}$$

(L2) $[x, x] \cap \omega_L \neq \emptyset, \quad \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \quad \forall x, y \in L$

Several generalizations of H_v -groups are defined, see [28], [30]. For example, the *h/v-groups* are defined where the *reproductivity of classes* is valid: $x\sigma(y) = \sigma(xy) = \sigma(x)y, \quad \forall x, y \in H$. Similarly, *h/v-rings*, *h/v-fields*, etc, are defined.

Moreover, several large classes of hyperoperations are defined, as well as the related hyperstructures, and studied deeply in [19], [20], [21], [22], [23], [28].

Definition 1.4 An H_v -structure is *very-thin* if all hyperoperations are operations except one, which has all hyperproducts singletons except one, which is a subset of cardinality more than one.

We can obtain a large number of very-thin hyperstructures from given structures by a procedure called *enlarging*. For example:

Theorem 1.5 In the ring $(\mathbb{Z}_n, +, \cdot)$, with $n = ms$ we enlarge the product only $0 \cdot m$ by setting $0 \otimes m = \{0, m\}$ and the rest results are the same. Then $(\mathbb{Z}_n, +, \otimes)/\gamma^* \cong (\mathbb{Z}_m, +, \cdot)$.

We can enlarge other products as well, as $2 \cdot m$ setting $2 \otimes m = \{2, m+2\}$, then the result remains the same. In this case 0 and 1 remain scalars.

Corollary. In $(\mathbb{Z}_n, +, \cdot)$, with $n=ps$, p prime, we enlarge only $0 \cdot p$ by $0 \otimes p = \{0, p\}$ and the rest remain the same. Then $(\mathbb{Z}_n, +, \otimes)$ is very-thin H_V -field.

Definition 1.6 Let $(G, \cdot)$ groupoid, $f: G \rightarrow G$ map, we define $\mathcal{A}$ -hyperoperation, by

$$x \hat{\otimes} y = \{f(x) \cdot y, x \cdot f(y)\}, \quad \forall x, y \in G.$$

Let $(A, +, \cdot)$ an algebra on F and $f: A \rightarrow A$, a map. The $\mathcal{A}$ -hyperoperation on the Lie bracket $[x, y] = xy - yx$, is defined by

$$x \hat{\otimes} y = \{f(x)y - f(y)x, f(x)y - yf(x), xf(y) - f(y)x, xf(y) - yf(x)\}.$$

Definition 1.7 Let $(G, \cdot)$ be groupoid, then $\forall P \subset G, P \neq \emptyset$, the P -hyperoperation are defined:

$$\underline{P}: x \underline{P} y = (xP)y \cup x(Py),$$

$$\underline{P}_r: x \underline{P}_r y = (xy)P \cup x(yP), \quad \underline{P}_l: x \underline{P}_l y = (Px)y \cup P(xy), \quad \forall x, y \in G$$

A generalization of P -hyperoperations, used in Santilli's isothory, is the following [3], [5], [30], [33]: Let $(G, \cdot)$ be abelian group and $P, |P| > 1$. We define a hyperoperation $(\times_P)$ by

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y \mid h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this P_e -hyperoperation. The hyperstructure $(G, \times_P)$ is abelian H_V -group.

2. THE ISO-HYPERNUMBERS AND LIE-SANTILLI ADMISSIBILITY

The Lie-Santilli theory on *isotopies* was born to solve Hadronic Mechanics problems [6], [7], [8], [9]. Santilli proposed a "lifting" of the n -dimensional trivial unit matrix of a normal theory into a new matrix. The corresponding in hyperstructures *e-isofields* introduced by Santilli and Vougiouklis in 1996 [10], [11], [24], [25], [29].

Definitions 2.1 $(H, \cdot)$ is called *e-hyperstructure* if it contains a unique scalar unit e . We assume that $\forall x$, there exists an inverse x^{-1} : $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

$(F, +, \cdot)$, where $(+)$ operation, $(\cdot)$ hyperoperation, is called *e-hyperfield* if:

$(F, +)$ is abelian group with unit 0, $(\cdot)$ is weak associative, $(\cdot)$ is weak distributive to $(+)$, 0 is absorbing: $0 \cdot x = x \cdot 0 = 0, \forall x \in F$, there is scalar unit 1: $1 \cdot x = x \cdot 1 = x, \forall x \in F$, and $\forall x \in F$ there is a unique inverse x^{-1} : $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*.

Definition 2.2 *The main e-Construction* [3], [5], [16], [17], [24], [25]. Let $(G, \cdot)$, be group, e unit, then we define in G , a large number of hyperoperations $(\otimes)$, where $(G, \otimes)$ is e-hypergroup, by:

$$x \otimes y = \{xy, g_1, g_2, \dots\}, \forall x, y \in G - \{e\}, g_1, g_2, \dots \in G - \{e\}$$

In Santilli's iso-theory, on a field $F=(F, +, \cdot)$, an isofield $\widehat{F} = \widehat{F}(\widehat{a}, \widehat{+}, \widehat{\times})$ has elements $\widehat{a} = a \times \widehat{1}$, *isonumbers*, $a \in F$, $\widehat{1}$ positive-defined outside F , with operations $\widehat{+}$, $\widehat{\times}$ where $\widehat{+}$ is the sum with additive unit 0, and $\widehat{\times}$ is a new product

$$\widehat{a} \widehat{\times} \widehat{b} := \widehat{a} \times \widehat{T} \times \widehat{b}, \quad \text{with } \widehat{1} = \widehat{T}^{-1}, \quad \forall \widehat{a}, \widehat{b} \in \widehat{F} \quad (i)$$

called *iso-product*, for which $\widehat{1}$ is the left and right unit of $\widehat{F}$, called *iso-unit*.

Construction 2.3 *General enlargement*. On a field $F=(F, +, \cdot)$ and the isofield $\widehat{F} = \widehat{F}(\widehat{a}, \widehat{+}, \widehat{\times})$ we replace in the results of the iso-product

$$\widehat{a} \widehat{\times} \widehat{b} = \widehat{a} \times \widehat{T} \times \widehat{b}, \quad \text{with } \widehat{1} = \widehat{T}^{-1} \quad (ii)$$

of the $\widehat{T}$ by a set, containing $\widehat{T}$, $\widehat{H}_{ab} = \{\widehat{T}, \widehat{x}_1, \widehat{x}_2, \dots\}$, where $\widehat{x}_1, \widehat{x}_2, \dots \in \widehat{F}$, $\forall \widehat{a} \widehat{\times} \widehat{b}$, $\widehat{a}, \widehat{b} \notin \{\widehat{0}, \widehat{1}\}$, $\widehat{x}_1, \widehat{x}_2, \dots \in \widehat{F} - \{\widehat{0}, \widehat{1}\}$. If one of $\widehat{a}$, $\widehat{b}$, or both, is equal to $\widehat{0}$ or $\widehat{1}$, then $\widehat{H}_{ab} = \{\widehat{T}\}$. Therefore, the new iso-hyperoperation is

$$\widehat{a} \widehat{\times} \widehat{b} = \widehat{a} \times \widehat{H}_{ab} \times \widehat{b} = \widehat{a} \times \{\widehat{T}, \widehat{x}_1, \widehat{x}_2, \dots\} \times \widehat{b}, \quad \forall \widehat{a}, \widehat{b} \in \widehat{F} \quad (iii)$$

$\widehat{F} = \widehat{F}(\widehat{a}, \widehat{+}, \widehat{\times})$ becomes *isoH_v-field*. The elements of $\widehat{F}$ are called *isonumbers*.

More important hyperoperations, are the ones where only for few pairs $(\widehat{a}, \widehat{b})$ the result is enlarged. Even more, if there exists only one pair $(\widehat{a}, \widehat{b})$ for which

$$\widehat{a} \widehat{\times} \widehat{b} = \widehat{a} \times \{\widehat{T}, \widehat{x}\} \times \widehat{b}, \quad \forall \widehat{a}, \widehat{b} \in \widehat{F}$$

then we obtain the *very-thin isoH_v-field* [3], [5], [6], [16], [17], [24], [28].

Construction 2.4 *The P-hyperoperation*. Consider an isofield $\widehat{F} = \widehat{F}(\widehat{a}, \widehat{+}, \widehat{\times})$, $\widehat{a} = a \times \widehat{1}$, $a \in F$, and $\widehat{1}$ is a positive-defined element F , where $\widehat{+}$ the sum and $\widehat{\times}$ the iso-product

$$\widehat{a} \widehat{\times} \widehat{b} := \widehat{a} \times \widehat{T} \times \widehat{b}, \quad \text{with } \widehat{1} = \widehat{T}^{-1}, \quad \forall \widehat{a}, \widehat{b} \in \widehat{F}$$

Take $\widehat{P} = \{\widehat{T}, \widehat{p}_1, \dots, \widehat{p}_s\}$, $\widehat{p}_1, \dots, \widehat{p}_s \in \widehat{F} - \{\widehat{0}, \widehat{1}\}$, define *iso-P-H_v-field*, $\widehat{F} = \widehat{F}(\widehat{a}, \widehat{+}, \widehat{\times}_P)$ where the hyperoperation $\widehat{\times}_P$ is defined by:

$$\widehat{a} \widehat{\times}_P \widehat{b} := \begin{cases} \widehat{a} \times \widehat{P} \times \widehat{b} = \{\widehat{a} \times \widehat{h} \times \widehat{b} \mid \widehat{h} \in \widehat{P}\} & \text{if } \widehat{a} \neq \widehat{1} \text{ and } \widehat{b} \neq \widehat{1} \\ \widehat{a} \times \widehat{T} \times \widehat{b} & \text{if } \widehat{a} = \widehat{1} \text{ or } \widehat{b} = \widehat{1} \end{cases}$$

The elements of $\hat{F}$ are called *iso-P-H_v-numbers*.

Example on $(Z_7, +, \cdot)$. In $\hat{Z}_7 = \hat{Z}_7(\hat{a}, \hat{+}, \hat{\times})$, denote $\hat{Z}_7 = \{\hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{4}, \hat{5}, \hat{6}\}$, the weak associative product hyperoperation is described by only one enlargement $\hat{2} \hat{\times} \hat{6} = \{\hat{2}, \hat{5}\}$.

We focus on *small non-degenerate H_v-fields* on $(Z_n, +, \cdot)$, which in Santilli's isothory, satisfy the conditions:

(1) *very-thin minimal*, (2) *weak commutative*, (3) *0,1 scalars*, (4) *unique inverses*

Thus, we enlarge the ring product by putting one more element. We cannot enlarge the result if it is 1 and we cannot put 1 in the enlargement.

The Lie-Santilli admissibility in H_v-structures is defined as follows [3], [5], [6], [8], [9], [10], [11], [24], [26], [27], [33]:

Definition 2.5 Let L be H_v-vector space on $(F, +, \cdot)$, $\varphi: F \rightarrow F/\gamma^*$, canonical, $\omega_F = \{x \in F: \varphi(x) = 0\}$, where 0 is zero of F/γ^* , ω_L the core of $\varphi': L \rightarrow L/\varepsilon^*$, 0 zero of L/ε^* . Take $R, S \in L$ then a *Lie-Santilli admissible hyperalgebra* is obtained by taking the hyperoperation Lie-bracket

$$[\cdot, \cdot]_{RS}: L \times L \rightarrow P(L): [x, y]_{RS} = (xR)y - (yS)x$$

Special cases are:

- (a) When only S is considered, then $[x, y]_S = xy - ySx$
- (b) When only R is considered, then $[x, y]_R = xRy - yx$

3. HELIX-HOPES AND HYPERMATRIX REPRESENTATIONS

Representation Theory on H_v-structures has already been introduced and studied in [5], [19], [22]. The Representations of H_v-groups can be, mainly, achieved by H_v-matrices.

Definitions 3.1 *H_v-matrix* is a matrix with entries elements of an H_v-field. The hyperproduct of two H_v-matrices $A = (a_{ij})$ and $B = (b_{ij})$, of type $m \times n$ and $n \times r$ respectively, is defined, in the usual manner,

$$A \cdot B = (a_{ij}) \cdot (b_{ij}) = \{C = (c_{ij}) \mid c_{ij} \in \bigoplus \Sigma a_{ik} \cdot b_{kj}\},$$

and it is a set of $m \times r$ H_v-matrices.

Let $(H, \cdot)$ be H_v-group, F be H_v-field, then it is called *H_v-matrix representation* on a set $M_F = \{(a_{ij}) \mid a_{ij} \in F\}$ any map

$$T: H \rightarrow M_F: h \mapsto T(h) \text{ such that } T(h_1 h_2) \cap T(h_1)T(h_2) \neq \emptyset, \forall h_1, h_2 \in H.$$

If $T(h_1h_2) \subset T(h_1)T(h_2)$, $\forall h_1, h_2 \in H$, then T is an *inclusion representation*. If $T(h_1h_2) = T(h_1)T(h_2) = \{T(h) \mid h \in h_1h_2\}$, $\forall h_1, h_2 \in H$, then T is a *good representation* and if T is one to one and good then it is a *faithful representation*.

The main theorem of representations is the following:

Theorem 3.2 A necessary condition to have an inclusion representation T of an H_V -group $(H, \cdot)$ by $n \times n$ H_V -matrices over the $(F, +, \cdot)$ is the following:

$\forall \beta^*(x)$, $x \in H$ there must exist $a_{ij} \in H$, $i, j \in \{1, \dots, n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, \dots, n\}\}$$

A hyperoperation on non-square ordinary matrices can be defined. We present the basic definitions on helix-hyperoperation [4], [12], [14], [15], [31].

Definitions 3.3 Let $A = (a_{ij}) \in M_{m \times n}$ be $m \times n$ matrix and $s, t \in \mathbb{N}$, $1 \leq s \leq m$, $1 \leq t \leq n$. We define the map *helix-st* by

$$\underline{st}: M_{m \times n} \rightarrow M_{s \times t}: A \rightarrow A \underline{st} = (\underline{a}_{ij}),$$

where $\underline{a}_{ij} = \{a_{i+\kappa s, j+\lambda t} \mid 1 \leq i \leq s, 1 \leq j \leq t, \kappa, \lambda \in \mathbb{N}, i+\kappa s \leq m, j+\lambda t \leq n\}$.

Definitions 3.4 (a) Let $A = (a_{ij}) \in M_{m \times n}$, $B = (b_{ij}) \in M_{u \times v}$, $s = \min(m, u)$, $t = \min(n, u)$. The hyperoperation *helix-sum*, is defined by

$$\oplus: M_{m \times n} \times M_{u \times v} \rightarrow P(M_{s \times t}): (A, B) \rightarrow A \oplus B = A \underline{st} + B \underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{s \times t},$$

Where $(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij} + b_{ij}) \mid a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}$.

(b) Let $A = (a_{ij}) \in M_{m \times n}$, $B = (b_{ij}) \in M_{u \times v}$, $s = \min(n, u)$. The *helix-product*, is defined by

$$\otimes: M_{m \times n} \times M_{u \times v} \rightarrow P(M_{m \times v}): (A, B) \rightarrow A \otimes B = A \underline{ms} \cdot B \underline{sv} = (\underline{a}_{ij}) \cdot (\underline{b}_{ij}) \subset M_{m \times v},$$

where $(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = (\sum a_{it} b_{tj}) \mid a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}$.

The helix-sum is commutative and the helix-product is weak associative.

The *helix-Lie Algebra* [16], [29], [33], is defined as well.

We study the matrices $M_{m \times n}$ where $m < n$. since we have analogous cases if $m > n$ and for $m = n$ we have the classical theory.

Several classes appropriate in helix-operations are defined [13], [15], [32].

Definition 3.5 Let $A = (a_{ij}) \in M_{m \times n}$ and $s, t \in \mathbb{N}$, $1 \leq s \leq m$, $1 \leq t \leq n$. We call *S-helix matrix of type sxt* a matrix A if the $A \underline{st}$ has on the diagonal entries which are not sets but elements. An S-helix matrix A , is called *S₀-helix matrix* if, moreover, the condition $a_{11} \dots a_{mm} \neq 0$, is valid.

Properties. The set of S-helix matrices $A = (a_{ij}) \in M_{m \times n}$ with $m < n$, is closed under helix-product. It has a helix-st unit matrix I_c , which is left scalar. The S_0 -helix matrices X have inverses X^{-1} , i.e. if I_c is the unit matrix, $I_c \in X \otimes X^{-1} \cap X^{-1} \otimes X$.

Example 3.6 A 6-dimensional representation with helix-hyperoperation, is the following: On real or complex numbers we consider all matrices of type

$$A = \begin{pmatrix} 1 & a & b & 1 & d & e \\ 0 & 1 & c & 0 & 1 & g \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

This set is closed under the helix hope:

$$A \otimes A = \begin{pmatrix} 1 & a & b & 1 & d & e \\ 0 & 1 & c & 0 & 1 & g \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \underline{a} & \underline{b} & 1 & \underline{d} & \underline{e} \\ 0 & 1 & \underline{c} & 0 & 1 & \underline{g} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & \{a+\underline{a}, d+\underline{a}\} & \{b+\underline{b}+ac, b+\underline{b}+dc, e+\underline{e}+ac, e+\underline{e}+dc\} & 1 & \{a+\underline{d}, d+\underline{d}\} & \{b+\underline{e}+ag, e+\underline{e}+ag, b+\underline{e}+dg, e+\underline{e}+dg\} \\ 0 & 1 & \{c+\underline{c}, g+\underline{c}\} & 0 & 1 & \{c+\underline{g}, g+\underline{g}\} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

So, the result is a set with 2^8 matrices.

In order to explain the way that helix-hyperoperation acts we claim that the helix-hyperoperation replaces some elements and shifts them together along with the corresponding elements, treating them in the same way, as elements of a set. One can say that this modulo-like procedure reminds us of the ‘repetition’ in teaching or the ‘motivos’ in music compositions.

Example 3.7 Consider the case of matrices of the type 4×7 on a field of real or complex numbers and on any finite field. Take

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{11} & x_{16} & x_{17} \\ 0 & x_{22} & x_{23} & x_{24} & 0 & x_{22} & x_{27} \\ 0 & 0 & x_{33} & x_{34} & 0 & 0 & x_{33} \\ 0 & 0 & 0 & x_{44} & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} & y_{11} & y_{16} & y_{17} \\ 0 & y_{22} & y_{23} & y_{24} & 0 & y_{22} & y_{27} \\ 0 & 0 & y_{33} & y_{34} & 0 & 0 & y_{33} \\ 0 & 0 & 0 & y_{44} & 0 & 0 & 0 \end{pmatrix}$$

We have

$$X \otimes Y = \begin{pmatrix} x_{11} & \{x_{12}, x_{16}\} & \{x_{13}, x_{17}\} & x_{14} \\ 0 & x_{22} & \{x_{23}, x_{27}\} & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix} \cdot \begin{pmatrix} y_{11} & y_{12} & y_{13} & y_{14} & y_{11} & y_{16} & y_{17} \\ 0 & y_{22} & y_{23} & y_{24} & 0 & y_{22} & y_{27} \\ 0 & 0 & y_{33} & y_{34} & 0 & 0 & y_{33} \\ 0 & 0 & 0 & y_{44} & 0 & 0 & 0 \end{pmatrix}$$

Then, denoting C_{ij} the ij entry of the result, we have

$$C_{11} = \{x_{11}y_{11}\}, \quad C_{12} = \{x_{11}y_{12} + \{x_{12}, x_{16}\}y_{22}\},$$

$$C_{13} = \{x_{11}y_{13} + \{x_{12}, x_{16}\}y_{23} + \{x_{13}, x_{17}\}y_{33}\},$$

$$\begin{aligned} C_{14} &= \{x_{11}y_{14} + \{x_{12}, x_{16}\}y_{24} + \{x_{13}, x_{17}\}y_{34} + x_{44}y_{44}\}, C_{15} = \{x_{11}y_{11}\}, \\ C_{16} &= \{x_{11}y_{16} + \{x_{12}, x_6\}y_{22}\}, C_{17} = \{x_{11}y_{17} + \{x_{12}, x_{16}\}y_{27} + \{x_8, x_{17}\}y_{33}\}, \\ C_{21} &= \{0\}, C_{22} = \{x_{22}y_{22}\}, C_{23} = \{x_{22}y_{23} + \{x_{23}, x_{27}\}y_{33}\}, \\ C_{24} &= \{x_{22}y_{24} + \{x_{23}, x_2\}y_{34} + x_{24}y_{44}\}, C_{25} = \{0\}, C_{26} = \{x_{22}y_{22}\}, \\ C_{27} &= \{x_{22}y_{27} + \{x_{23}, x_{27}\}y_{33}\}, C_{31} = \{0\}, C_{32} = \{0\}, C_{33} = \{x_{33}y_{33}\}, \\ C_{34} &= \{x_{33}y_{34} + x_{34}y_{44}\}, C_{35} = \{0\}, E_{36} = \{0\}, C_{37} = \{x_{33}y_{33}\}, C_{41} = \{0\}, C_{42} = \{0\}, \\ C_{43} &= \{0\}, C_{44} = \{x_{44}y_{44}\}, C_{45} = \{0\}, C_{46} = \{0\}, C_{47} = \{0\}. \end{aligned}$$

Therefore, the helix product is a set with cardinality up to 2^{11} .

Consider the minimal non-degenerate H_V -fields on the finite ring $(\mathbb{Z}_n, +, \cdot)$, which in isothory satisfy the Santilli's 4 conditions [4], [7], [9] [1], [2], [3], [3].

Example 3.8 Take in the above, $n=10$ and in this case the only hyper-result is $2 \otimes 4 = \{3, 8\}$. Consequently the fundamental classes are: $[0] = \{0\}$, $[1] = \{1, 6\}$, $[2] = \{2, 7\}$, $[3] = \{3, 8\}$, $[4] = \{4, 9\}$ and $(\mathbb{Z}_{10}, +, \otimes)/\gamma^* \cong (\mathbb{Z}_5, +, \cdot)$.

Consider the 4×7 S-helix matrices on the above H_V -field and take the

$$A = \begin{pmatrix} 7 & 2 & 2 & 6 & 7 & 7 & 4 \\ 0 & 1 & 3 & 8 & 0 & 1 & 4 \\ 0 & 0 & 3 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 9 & 3 & 5 & 4 & 6 & 7 \\ 0 & 1 & 0 & 4 & 0 & 1 & 2 \\ 0 & 0 & 8 & 4 & 0 & 0 & 8 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \end{pmatrix}$$

then, we have

$$\begin{aligned} A \otimes B &= \begin{pmatrix} 7 & \{2,7\} & \{2,4\} & 6 \\ 0 & 1 & \{3,4\} & 8 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 9 & 3 & 5 & 4 & 6 & 7 \\ 0 & 1 & 0 & 4 & 0 & 1 & 2 \\ 0 & 0 & 8 & 4 & 0 & 0 & 8 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} 8 & 3 + \{2, 7\} & 1 + 8\{2,4\} & 5 + \{2,7\} & 4 + \{2,4\} & 4 + 8 & 8 & 2 + \{2,7\} & 9 + \{2, 7\} + \{2,4\} & 8 \\ 0 & 1 & 8\{3, 4\} & 4 + \{3,4\} & 4 + 4 & 0 & 1 & 2 + \{3,4\} & 8 \\ 0 & 0 & 4 & 2 + 6 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \\ &= \begin{pmatrix} 8 & \{0, 3\} & \{3, 7\} & \{2,4,7\} & 9 & 8 & \{4, 9\} & \{5, 9\} \\ 0 & 1 & \{2,4\} & \{0,4\} & 0 & 1 & \{4,6\} \\ 0 & 0 & 4 & 8 & 0 & 0 & 4 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

For the powers of the S-helix matrix A we have

$$A^2 = \begin{pmatrix} 7 & \{2,7\} & \{2,4\} & 6 \\ 0 & 1 & \{3,4\} & 8 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 7 & 2 & 2 & 6 & 7 & 7 & 4 \\ 0 & 1 & 3 & 8 & 0 & 1 & 4 \\ 0 & 0 & 3 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 9 & 4+\{2,7\} & 4+\{2,7\}3+\{2,4\}3 & 2+\{2,7\}8+\{2,4\}2+2 & 9 & 9+\{2,7\} & 8+\{2,7\}4+\{2,4\}3 \\ 0 & 1 & 3+\{3,4\}3 & 8+\{3,4\}2+6 & 0 & 1 & 4+\{3,4\}3 \\ 0 & 0 & 9 & 6+4 & 0 & 0 & 9 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 9 & \{1,6\} & \{1,2,6,7\} & \{4,8\} & 9 & \{1,6\} & \{2,3,7,8\} \\ 0 & 1 & \{2,5\} & \{0,2\} & 0 & 1 & \{3,6\} \\ 0 & 0 & 9 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \end{pmatrix}$$

For the 3rd power the cyclic hyperproduct: $A^3 = (A^2 A \otimes A) \cup (A \otimes A^2)$ is

$$A^2 \otimes A = \begin{pmatrix} 9 & \{1,6\} & \{1,2,6,7\} & \{4,8\} & 9 & \{1,6\} & \{2,3,7,8\} \\ 0 & 1 & \{2,5\} & \{0,2\} & 0 & 1 & \{3,6\} \\ 0 & 0 & 9 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 7 & 2 & 2 & 6 & 7 & 7 & 4 \\ 0 & 1 & 3 & 8 & 0 & 1 & 4 \\ 0 & 0 & 3 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 9 & \{1,6\} & \{1,2,3,6,7,8\} & \{4,8\} \\ 0 & 1 & \{2,3,5,6\} & \{0,2\} \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} 7 & 2 & 2 & 6 & 7 & 7 & 4 \\ 0 & 1 & 3 & 8 & 0 & 1 & 4 \\ 0 & 0 & 3 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 3 & \{4,9\} & \{0,2,4,5,7,9\} & \{4,6,8\} & 3 & \{4,9\} & \{1,3,4,6,8,9\} \\ 0 & 1 & \{1,2,8,9\} & \{0,2,4,6,8\} & 0 & 1 & \{0,2,3,9\} \\ 0 & 0 & 7 & 8 & 0 & 0 & 7 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 \end{pmatrix}$$

and, on the other hand, we have

$$A \otimes A^2 = \begin{pmatrix} 7 & 2 & 2 & 6 & 7 & 7 & 4 \\ 0 & 1 & 3 & 8 & 0 & 1 & 4 \\ 0 & 0 & 3 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 9 & \{1,6\} & \{1,2,6,7\} & \{4,8\} & 9 & \{1,6\} & \{2,3,7,8\} \\ 0 & 1 & \{2,5\} & \{0,2\} & 0 & 1 & \{3,6\} \\ 0 & 0 & 9 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 7 & \{2,7\} & \{2,4\} & 6 \\ 0 & 1 & \{3,4\} & 8 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 9 & \{1,6\} & \{3,7\} & \{1,2,6,7\} & 9 & \{1,6\} & \{2,3,7,8\} \\ 0 & 1 & \{2,5\} & \{0,2\} & 0 & 1 & \{3,6\} \\ 0 & 0 & 9 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 3 & \{4,9\} & \{0,1,2,3,4,5,7,9\} & \{0,1,2,3,5,6,7,8\} & 3 & \{4,9\} & \mathbf{Z}_{10} \\ 0 & 1 & \{1,2,8,9\} & \{2,4\} & 0 & 1 & \{0,2,3,9\} \\ 0 & 0 & 7 & \{3,8\} & 0 & 0 & 7 \\ 0 & 0 & 0 & \{3,8\} & 0 & 0 & 0 \end{pmatrix}$$

Therefore,

$$A^3 = (A^2 \otimes A) \cup (A \otimes A^2) = \begin{pmatrix} 3 & \{4,9\} & \mathbf{Z}_{10} - \{6,8\} & \mathbf{Z}_{10} - \{9\} & 3 & \{4,9\} & \mathbf{Z}_{10} \\ 0 & 1 & \{1,2,8,9\} & \{0,2,4,6,8\} & 0 & 1 & \{0,2,3,9\} \\ 0 & 0 & 7 & \{3,8\} & 0 & 0 & 7 \\ 0 & 0 & 0 & \{3,8\} & 0 & 0 & 0 \end{pmatrix}$$

Remarks. From the fact that the number of H_v -structures defined on a set, is very big, one can ask for more properties and restrictions to reduce this number. Thus, the minimal H_v -fields obtained from rings, enlarging only the result of two elements a, b , by setting $a \otimes b = \{ab, c\}$, where $ab \neq c$. If we want the diagonal products to be singletons, then a or b could not be used.

4. APPLICATIONS

Example 4.1 We fix the real 4×7 S_0 -helix matrices

$$R = \begin{pmatrix} 1 & 0 & 2 & 2 & 1 & 0 & 2 \\ 0 & 5 & 1 & 3 & 0 & 5 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 2 & 0 & 3 & 1 & 2 & 3 \\ 0 & 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 5 & 4 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \end{pmatrix}$$

then, for the S_0 -helix matrices

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 & 0 & 1 & 4 \\ 0 & 0 & 5 & 2 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 3 & 3 & 5 & 5 & 3 & 3 \\ 0 & 1 & 3 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

we have

$$[A, B]_{RS} = (A \otimes R) \otimes B - (B \otimes S) \otimes A =$$

$$\left(\begin{pmatrix} 1 & 1 & \{2,3\} & 2 \\ 0 & 1 & \{1,4\} & 3 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 2 & 1 & 0 & 2 \\ 0 & 5 & 1 & 3 & 0 & 5 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \right) \otimes B -$$

$$- \left(\begin{pmatrix} 5 & 3 & 3 & 5 \\ 0 & 1 & \{2,3\} & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 3 & 1 & 2 & 3 \\ 0 & 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 5 & 4 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \end{pmatrix} \right) \otimes A =$$

Then, the final result is the set

$$\begin{pmatrix} 0 & -10 & \{22,23,24\} - \{98,173\} & \{29,31,33,35\} - \{143,173\} & 0 & -10 & \{17,18,19\} - \{142,217\} \\ 0 & 4 & \{16,17,19,20\} - \{51,76\} & \{15,17,21,23,27,29\} - \{40,44,50,54\} & 0 & 4 & \{11,12,14,15\} - \{54,79\} \\ 0 & 0 & -20 & -4 & 0 & 0 & -20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, let us see the same example but on the ring $\mathbb{Z}_6$. Then, we have

$$[A, B]_{RS} = (A \otimes R) \otimes B - (B \otimes S) \otimes A =$$

$$\begin{pmatrix} 5 & 2 & \{0,4,5\} & \{1,3,5\} & 5 & 2 & \{0,1,5\} \\ 0 & 5 & \{1,2,4,5\} & \{3,5\} & 0 & 5 & \{0,2,3,5\} \\ 0 & 0 & 5 & 2 & 0 & 0 & 5 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 5 & 0 & \{2,5\} & 5 & 5 & 0 & \{2,4\} \\ 0 & 1 & \{3,4\} & \{0,2,4\} & 0 & 1 & \{0,1\} \\ 2 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 5 & 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 2 & \mathbb{Z}_6 & \{0,2,4\} & 0 & 2 & \{1,2,3,4,5\} \\ 0 & 4 & \mathbb{Z}_6 & \{1,3,5\} & 0 & 4 & \mathbb{Z}_6 \\ 0 & 0 & 4 & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In representation theory of hyperstructures, the helix-products can also be used on H_V -fields instead of fields.

Example 4.2 Consider the *minimal non-degenerate H_V -fields* on $(\mathbb{Z}_n, +, \cdot)$, which in isothory satisfy the above 4 conditions, see also Corollary of Theorem 1. 5. Take, as in Example 3. 8 $n=10$ and the only hyper-result is $2 \otimes 4 = \{3, 8\}$.

We fix the 3×6 S_0 -helix matrices on the above H_V -field

$$R = \begin{pmatrix} 1 & 2 & 1 & 1 & 7 & 6 \\ 0 & 3 & 1 & 0 & 3 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{pmatrix}, S = \begin{pmatrix} 1 & 2 & 0 & 1 & 7 & 0 \\ 0 & 1 & 6 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 0 & 3 \end{pmatrix}$$

then, for the S_0 -helix matrices

$$A = \begin{pmatrix} 1 & 1 & 6 & 1 & 1 & 1 \\ 0 & 1 & 9 & 0 & 1 & 4 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 & 5 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 & 1 & 8 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

we have

$$\begin{aligned} [A, B]_{RS} &= (A \otimes R) \otimes B - (B \otimes S) \otimes A = \\ &= \left(\begin{pmatrix} 1 & 1 & \{1,6\} \\ 0 & 1 & \{4,9\} \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 1 & 7 & 6 \\ 0 & 3 & 1 & 0 & 3 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{pmatrix} \right) \otimes B \\ &\quad - \left(\begin{pmatrix} 2 & 3 & \{0,5\} \\ 0 & 1 & \{3,8\} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 & 1 & 7 & 0 \\ 0 & 1 & 6 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & 0 & 3 \end{pmatrix} \right) \otimes A = \\ &= \begin{pmatrix} 1 & \{0,5\} & \{4,9\} \\ 0 & 3 & 9 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 & 5 & 2 & 3 & 0 \\ 0 & 1 & 3 & 0 & 1 & 8 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \\ &\quad - \begin{pmatrix} 2 & 7 & \{1,6\} \\ 0 & 1 & \{0,5\} \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 6 & 1 & 1 & 1 \\ 0 & 1 & 9 & 0 & 1 & 4 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{pmatrix} = \\ &= \begin{pmatrix} 2 & \{3,8\} & \{4,9\} & 2 & \{3,8\} & \{4,9\} \\ 0 & 3 & 8 & 0 & 3 & 3 \\ 0 & 0 & 4 & 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 9 & 7 & 2 & 9 & 2 \\ 0 & 1 & 9 & 0 & 1 & 4 \\ 0 & 0 & 6 & 0 & 0 & 6 \end{pmatrix} = \\ &= \begin{pmatrix} 0 & \{4,9\} & \{2,7\} & 0 & \{4,9\} & \{2,7\} \\ 0 & 2 & 9 & 0 & 2 & 9 \\ 0 & 0 & 8 & 0 & 0 & 8 \end{pmatrix} \end{aligned}$$

Example 4.3 Consider, again, the *minimal non-degenerate H_v -field* on $(\mathbb{Z}_{10}, +, \cdot)$, which in isothecy satisfying the 4 conditions and where the only hyper-result is $2 \otimes 4 = \{3, 8\}$.

We fix the 4×8 S_0 -helix matrices on the above H_v -field

$$R = \begin{pmatrix} 1 & 3 & 0 & 3 & 1 & 3 & 5 & 8 \\ 0 & 2 & 1 & 4 & 0 & 2 & 1 & 9 \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 6 \end{pmatrix}, \quad S = \begin{pmatrix} 4 & 1 & 5 & 0 & 4 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 \end{pmatrix}$$

then, for the S_0 -helix matrices

$$A = \begin{pmatrix} 2 & 2 & 1 & 6 & 2 & 2 & 6 & 6 \\ 0 & 1 & 3 & 2 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 4 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 & 1 & 0 & 3 & 2 & 6 & 0 \\ 0 & 2 & 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \end{pmatrix}$$

we have

$$\begin{aligned} [A, B]_{\mathfrak{A}} &= (A \otimes R) \otimes B - (B \otimes S) \otimes A = \\ &= \left(\begin{pmatrix} 2 & 2 & \{1,6\} & 6 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & \{4,9\} \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 & 3 & 1 & 3 & 5 & 8 \\ 0 & 2 & 1 & 4 & 0 & 2 & 1 & 9 \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 6 \end{pmatrix} \right) \otimes B \\ &\quad - \left(\begin{pmatrix} 3 & 2 & \{1,6\} & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & \{0,5\} \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 & 1 & 5 & 0 & 4 & 1 & 0 & 0 \\ 0 & 4 & 5 & 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 \end{pmatrix} \right) \otimes A = \\ &= \begin{pmatrix} 2 & 0 & 4 & \{0,5\} \\ 0 & 2 & 7 & \{1,6\} \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 & 0 & 3 & 2 & 6 & 0 \\ 0 & 2 & 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \end{pmatrix} \\ &\quad - \begin{pmatrix} 2 & \{1,6\} & \{1,6\} & \{3,8\} \\ 0 & \{3,8\} & 1 & \{3,8\} \\ 0 & 0 & 1 & \{3,8\} \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 & 6 & 2 & 2 & 6 & 6 \\ 0 & 1 & 3 & 2 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 4 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \end{pmatrix} = \end{aligned}$$

$$\begin{pmatrix} 6 & 4 & 6 & 0 & 6 & 4 & 6 & 0 \\ 0 & 4 & 9 & 2 & 0 & 4 & 9 & 7 \\ 0 & 0 & 2 & 8 & 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 4 & \{0,5\} & \{1,6\} & 4 & 4 & \{0,5\} & \{1,6\} & \{4,9\} \\ 0 & \{3,8\} & \{0,5\} & 6 & 0 & \{3,8\} & \{0,5\} & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & \{4,9\} & \{0,5\} & 6 & 2 & \{4,9\} & \{0,5\} & \{1,6\} \\ 0 & \{1,6\} & \{4,9\} & 6 & 0 & \{1,6\} & \{4,9\} & 6 \\ 0 & 0 & 1 & 8 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \end{pmatrix}$$

On the above example we remark the followings: Applying the helix-st map in helix-hyperoperation, if all the elements which we put in the same entry, belong on the same fundamental class, then the hyper-sums and hyper-products guarantee that they belong on the same fundamental classes. Thus, according Theorem 3. 2, the subset of S_0 -matrices for which the helix-st map have entries of the same fundamental class, is a closed under the helix-hyperoperation.

5. CONCLUSION

When dealing with a great amount of data or any type of information, we have serious problem of managing all this data so as that once embedded in our minds, then to be accessible, easy to teach or transfer to another person-receiver. The Mathematical Model Helix might give an easily applicable method- solution in cases like that. In this way, every element is present and every element maintains its independence, appears self-contained and easily available. More specifically here, we study the helix-hopes on minimal H_v -fields in order to shorten the results. Apart to the main use of the helix-hyperoperation as model one can also use it in other sciences. We present applications of the helix case, as a mathematical model in the Lie-Santilli's admissible Theory, because it allows us to use non-square ordinary matrices.

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ON RAISED H_v -FIELDS

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Abstract

The H_v -structures defined on a set admits a partial order and has a numerous of applications in mathematics and other sciences. The weak axioms, which the H_v -structures satisfy, is the main reason that they can be used even as organized device. On the other side, in applications one needs small results, thus, the researchers ask for H_v -structures which are near to the classical structures. For this reason, we focus on the *raised H_v -fields*.

Keywords: *Hyperstructure, hope, H_v -group H_v -ring, H_v -field.*

1. INTRODUCTION

Hyperoperation (abbreviate *hope*) in a set H , is called any map $(\cdot)$:

$$\cdot: H \times H \rightarrow P(H) - \{\emptyset\}.$$

Hyperstructure is called any set equipped with at least one hope. The largest class of hyperstructures is the *H_v-structures*, named after T. Vougiouklis, introduced in 1990, in 4th AHA Congress [9], [10], [5]. They satisfy the *weak axioms* where, the equality is replaced by the non-empty intersection. So, in a set with a hope $(H, \cdot)$, *weak associative* means

$$(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H,$$

and *weak commutative*

$$xy \cap yx \neq \emptyset, \forall x, y \in H.$$

$(H, \cdot)$ is called *H_v-semigroup* if is weak associative and it is called *H_v-group* if, moreover, it is reproductive: $xH = Hx = H, \forall x \in H$.

Connection of structures with hyperstructures: The quotient of a group by an invariant subgroup, is a group; the quotient of a group by a subgroup, is a hypergroup; the quotient of a group by a partition, is an *H_v-group*.

In *H_v-semigroups*, the *powers* are defined using the *circle hope* which is the union of hyperproducts, with all patterns of parentheses on them. The *period* of a generator could be less than the cardinality of the *H_v-semigroup*. If one power covers the underline set, then we have a *single-power cyclic H_v-semigroup*.

$(R, +, \cdot)$ is called *H_v-ring* if $(+)$ and $(\cdot)$ are weak associative, $(+)$ is reproductive, and $(\cdot)$ is *weak distributive* to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \emptyset, (x+y)z \cap (xz+yz) \neq \emptyset, \forall x, y, z \in R.$$

Let $(R, +, \cdot)$ be *H_v-ring*, then a weak commutative *H_v-group* $(M, +)$ is called *H_v-module* over R , if there is a hope

$$\cdot: R \times M \rightarrow P(M) - \{\emptyset\}: (a, x) \mapsto ax$$

such that, $\forall a, b \in R, \forall x, y \in M$, we have

$$a(x+y) \cap (ax+ay) \neq \emptyset, (a+b)x \cap (ax+bx) \neq \emptyset, (ab)x \cap a(bx) \neq \emptyset.$$

The *H_v-vector space* is defined if we take an *H_v-field*, defined later, instead of an *H_v-ring*.

Definitions and applications on H_v -structures one can see in books and papers as [1], [2], [3], [5], [10], [11], [12], [15].

Given H_v -semigroups $(H, \cdot)$, $(H, *)$ on the same set, $(\cdot)$ is *smaller* than $(*)$, $(*)$ *greater* than $(\cdot)$, if there is an automorphism of $(H, *)$, for which $xy \subset f(x*y)$, $\forall x, y \in H$. Then, $(H, *)$ *contains* $(H, \cdot)$. *Minimal* is an H_v -group if it contains not any H_v -group.

The Little Theorem. Hopes containing weak associative or weak commutative hopes, are weak associative or weak commutative, respectively, as well.

A class of H_v -structures, introduced in [8], [16], [17], is the following:

Definition 1.1 An H_v -structure is *very-thin* if, and only if, all hopes are operations except one, with all results singletons except only one, which is a subset of cardinality more than one.

Some large classes of H_v -structures are the following [10], [13]:

Definition 1.2 Let $(G, \cdot)$ groupoid (or hypergroupoid), $f: G \rightarrow G$ map. Define the ∂ -hope (∂) , by

$$x\partial y = \{f(x) \cdot y, x \cdot f(y)\}, \quad (\text{or } x\partial y = (f(x) \cdot y) \cup (x \cdot f(y))), \quad \forall x, y \in G.$$

If $(\cdot)$ is commutative or weak, then (∂) is commutative or weak, respectively.

Definition 1.3 Let $(G, \cdot)$ groupoid, then $\forall P \subset G, P \neq \emptyset$, we define the P -hopes:

$\underline{P}$: $x\underline{P}y = (xP)y \cup x(Py)$, $\underline{P}_r$: $x\underline{P}_ry = (xy)P \cup x(yP)$, $\underline{P}_l$: $x\underline{P}_ly = (Px)y \cup P(xy)$, $\forall x, y \in G$.
 $(G, \underline{P})$, $(G, \underline{P}_r)$, $(G, \underline{P}_l)$ are called P -hyperstructures. If $(G, \cdot)$ semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is semihypergroup.

A generalization of P -hopes is the following [5]:

Definition 1.4 Let $(G, \cdot)$ be abelian group, $P \subset G$, $|P| > 1$. We define the P_e -hope $(\times_P)$ by

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y \mid h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e. \end{cases}$$

$(G, \times_P)$ is an abelian H_v -group.

2. FUNDAMENTAL RELATIONS AND H_V -FIELDS

Hyperstructures are connected with structures through the fundamental relations. M. Koskas, in 1970, introduced the relation β^* in hypergroups. In 1990, T. Vougiouklis introduced the relations γ^* and ε^* , giving a completely different proof of the main theorem on them, and named all these relations *fundamental*. Thus, the fundamental relations β^* , γ^* and ε^* , are defined, in H_V -groups, H_V -rings and H_V -vector spaces, respectively. One can see definitions, results and applications in [5], [9], [10], [11], [12], [14], [15], [16].

Definition 2.1 The *fundamental relations* β^* , γ^* , and ε^* in H_V -groups, H_V -rings, and H_V -vector spaces, respectively, are the smallest equivalences so that the quotient would be group, ring, and vector spaces, respectively.

Let $(G, \cdot)$ be group and R any partition in G , then $(G/R, \cdot)$ is H_V -group and $(G/R, \cdot)/\beta^*$ is the fundamental group.

The main theorems are the analogous to the following, which we prove here, in sort, for H_V -rings in order to define the H_V -field:

Theorem 2.2 Let $(R, +, \cdot)$ be H_V -ring, denote U the set of all finite polynomials of R . We define the relation γ in R by

$$x\gamma y \text{ if and only if } \{x, y\} \subset u, \text{ where } u \in U,$$

then, γ^* is the transitive closure of γ .

Proof. Let γ the transitive closure of γ , $\gamma(a)$ the class of a . In R/γ $(\oplus)$, $(\otimes)$ are defined as usual:

$$\gamma(a) \oplus \gamma(b) = \{\gamma(c) : c \in \gamma(a) + \gamma(b)\}, \quad \gamma(a) \otimes \gamma(b) = \{\gamma(d) : d \in \gamma(a) \cdot \gamma(b)\}, \quad \forall a, b \in R.$$

Take $a' \in \gamma(a)$, $b' \in \gamma(b)$. Then, we have

$$a' \gamma a \text{ iff } \exists x_1, \dots, x_{m+1} \text{ where } x_1 = a', x_{m+1} = a, u_1, \dots, u_m \in U, \{x_i, x_{i+1}\} \subset u_i, i=1, \dots, m,$$

and

$$b' \gamma b \text{ iff } \exists y_1, \dots, y_{n+1} \text{ where } y_1 = b', y_{n+1} = b, v_1, \dots, v_n \in U, \{y_j, y_{j+1}\} \subset v_j, j=1, \dots, n.$$

We obtain $\{x_i, x_{i+1}\} + y_1 \subset u_i + v_1, i=1, \dots, m-1, x_{m+1} + \{y_j, y_{j+1}\} \subset u_m + v_j, j=1, \dots, n.$

The $u_i + v_1 = t_i, i=1, \dots, m-1, u_m + v_j = t_{m+j-1}, j=1, \dots, n$, are polynomials:

$$t_k \in U, \quad \forall k \in \{1, \dots, m+n-1\}.$$

Pick up $z_1, \dots, z_{m+n}$ such that $z_i \in x_i + y_1$, $i=1, \dots, n$, $z_{m+j} \in x_{m+1} + y_{j+1}$, $j=1, \dots, n$, then, we obtain $\{z_k, z_{k+1}\} \subset t_k$, $k=1, \dots, m+n-1$.

Consequently, any $z_1 \in x_1 + y_1 = a' + b'$ is γ equivalent to any $z_{m+n} \in x_{m+1} + y_{n+1} = a + b$. So, $\gamma(a) \oplus \gamma(b)$ is singleton and we write $\gamma(a) \oplus \gamma(b) = \gamma(c)$, $\forall c \in \gamma(a) + \gamma(b)$.

Similarly, $\gamma(a) \otimes \gamma(b) = \gamma(d)$, $\forall d \in \gamma(a) \cdot \gamma(b)$.

Weak associativity and weak distributivity on R guarantee that associativity and distributivity are valid in R/γ^* . Therefore, R/γ^* is a ring.

Take σ equivalence in R , so that R/σ is ring, $\sigma(a)$ class of a . Then $\sigma(a) \oplus \sigma(b)$, $\sigma(a) \otimes \sigma(b)$ are singletons, thus, $\forall a, b \in R$, we have

$$\sigma(a) \oplus \sigma(b) = \sigma(c), \forall c \in \sigma(a) + \sigma(b) \text{ and } \sigma(a) \otimes \sigma(b) = \sigma(d), \forall d \in \sigma(a) \cdot \sigma(b).$$

Therefore, we have, $\forall a, b \in R$ and $A \subset \sigma(a)$, $B \subset \sigma(b)$,

$$\sigma(a) \oplus \sigma(b) = \sigma(a+b) = \sigma(A+B), \quad \sigma(a) \otimes \sigma(b) = \sigma(ab) = \sigma(A \cdot B).$$

By induction, we extend the relations to finite sums and products. So, $\forall u \in U$, we have $\sigma(x) = \sigma(u)$, $\forall x \in u$. Consequently, $x \in \gamma(a)$ implies $x \in \sigma(a)$, $\forall x \in R$.

But σ is transitively closed, so $x \in \gamma(x)$ implies $x \in \sigma(a)$.

Therefore, γ is the smallest equivalence such that R/γ is a ring and $\gamma = \gamma^*$. ■

An element is called *single* if its fundamental class is a singleton.

Fundamental relations are used for general definitions. So, we have [9], [7]:

Definition 2.3 An H_v -ring $(R, +, \cdot)$ is called *H_v -field* if R/γ^* is a field.

Definition 2.4 $(L, +)$ H_v -vector space on $(F, +, \cdot)$; $\phi: F \rightarrow F/\gamma^*$; $\omega_F = \{x \in F: \phi(x) = 0\}$ core; 0 zero of F/γ^* ; ω_L core of $\phi': L \rightarrow L/\varepsilon^*$; 0 zero of L/ε^* . The *bracket-hope*

$$[\cdot, \cdot]: L \times L \rightarrow P(L): (x, y) \mapsto [x, y]$$

gives is an *H_v -Lie algebra* over F if the following satisfied:

(L1) The bracket-hope is bilinear: $\forall x, x_1, x_2, y, y_1, y_2 \in L$, $\forall \lambda_1, \lambda_2 \in F$

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \emptyset, \quad [x, \lambda_1 y_1 + \lambda_2 y] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \emptyset,$$

(L2) $[x, x] \cap \omega_L \neq \emptyset$, $\forall x \in L$,

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset$, $\forall x, y, z \in L$.

Definition 2.5 The H_v -semigroup $(H, \cdot)$ is called *h/v-group* if H/β^* is a group.

The *h/v-group* is an H_v -group, where a *reproductive of classes*, is valid.

The *uniting elements* method, Corsini & Vougiouklis, is the following [2], [10]: Let G a structure where a property d , defined by equations, is not valid. Take a partition in G putting in the same class, all pairs caused the non-validity of d . G/d is H_v -structure and $(G/d)/\beta^*$ is a structure for which d , is valid.

Uniting elements, in two properties, in different order, the result is the same.

Theorem 2.6 Let $(H, \cdot)$ be H_v -group. If H/β^* is not commutative or is not cyclic, then $(H, \cdot)$ is not weak commutative or cyclic, respectively.

Proof. Immediate, since if $(H, \cdot)$ is weak commutative or cyclic then its fundamental group H/β^* is commutative or cyclic, respectively. ■

3. RAISED H_v -FIELDS

We simplify a definition, of the class of very-thin H_v -fields, given in [17] as follows:

Definition 3.1 An H_v -field is called *raised* if it is obtained from a ring, adding exactly one element of the underline set to only one result, such that, the fundamental structure is a field.

With the uniting elements procedure and raising operations or hopes we obtain stricter structures or H_v -structures.

We consider, now, the *raised H_v -fields* obtained by classical fields or rings.

Theorem 3.2 In a field $(F, +, \cdot)$, we raise only a product $a \cdot b$, by $a \otimes b = \{a \cdot b, c\}$, where $c \neq a \cdot b$. Then, we obtain the degenerate H_v -field $(F, +, \otimes)/\gamma^* \cong \{0\}$.

Proof. Take any $x \in F - \{0\}$, then from $a \otimes b = \{ab, c\}$ we obtain

$$(a \otimes b) \cdot ab = \{0, c \cdot ab\} \text{ and then } (x(c \cdot ab)^{-1}) \cdot ((a \otimes b) \cdot ab) = \{0, x\}$$

thus, $0 \gamma x$, $\forall x \in F - \{0\}$. Which means that every x is in the same fundamental class with 0. Therefore, $(F, +, \otimes)/\gamma^* \cong \{0\}$. ■

Theorem 3.3 In a field $(F, +, \cdot)$, we raise only a sum $a + b$, by $a \oplus b = \{a + b, c\}$, where $c \neq a + b$. Then, we obtain the degenerate H_v -field $(F, \oplus, \cdot)/\gamma^* \cong \{0\}$.

Proof. Take any $x \in F - \{0\}$, then from $a \oplus b = \{a+b, c\}$ we obtain

$$(a \oplus b) - (a+b) = \{0, c-(a+b)\} \text{ and then } [x(c-(a+b))]^{-1} \cdot [(a \oplus b) - (a+b)] = \{0, x\}$$

thus, $0 \gamma x, \forall x \in F - \{0\}$. Which means that every x is in the same fundamental class with 0. Therefore, $(F, \oplus, \cdot) / \gamma^* \cong \{0\}$. ■

The above two theorems state that there is no non-degenerate H_V -field obtained by a field raising any sum or product.

Theorem 3.4 In integers $(\mathbb{Z}, +, \cdot)$, fix $m > 1$, raise the result $a \cdot b$ of $a, b \in \mathbb{Z} - \{0, 1\}$, setting $a \otimes b = \{a \cdot b, a \cdot b + m\}$. Then, $(\mathbb{Z}, +, \otimes)$ is H_V -ring, with fundamental ring

$$(\mathbb{Z}, +, \otimes) / \gamma^* \cong (\mathbb{Z}_m, +, \cdot).$$

If $m=p$, prime, then $(\mathbb{Z}, +, \otimes)$ is a raised H_V -field, with finite fundamental field, where 0 and 1 are scalars.

The simplest case is when we raise only $0 \cdot m$ by setting $0 \otimes m = \{0, m\}$.

Proof. The expressions of sums and products which contain more than one element are the ones which have at least one time $a \otimes b$. Adding to $a \otimes b$ the number 1, k times, we have $k + a \otimes b = \{k + a \cdot b, k + a \cdot b + m\}$, the mod m equivalence classes. Adding to $a \otimes b$ the number m , s times, we have

$$a \otimes b + sm = \{a \cdot b + sm, a \cdot b + (s+1)m\},$$

the mod m equivalent elements. Moreover, adding or multiplying elements of the same class the results are remaining in one class, obtained by using only the representatives. Thus, the γ^* -classes form a ring isomorphic to $(\mathbb{Z}_m, +, \cdot)$.

If $m=p$, prime, then $(\mathbb{Z}, +, \otimes)$ is a raised H_V -field, because $(\mathbb{Z}_p, +, \cdot)$, is a finite field, where, obviously 0 and 1 are scalars. ■

Theorem 3.5 In the ring $(\mathbb{Z}_n, +, \cdot)$, with $n=ms$, and $m, s > 1$, we raise only the result $a \cdot b$ of fixed $a, b \in \mathbb{Z}_n - \{0, 1\}$, by setting $a \otimes b = \{a \cdot b, a \cdot b + m\}$. Then, $(\mathbb{Z}_n, +, \otimes)$ is very-thin H_V -ring, where 0, 1 are scalars and the fundamental ring is

$$(\mathbb{Z}_n, +, \otimes) / \gamma^* \cong (\mathbb{Z}_m, +, \cdot).$$

If $m=p$, prime, then $(\mathbb{Z}_n, +, \otimes)$ is a raised H_V -field.

The simplest case is when we raise only $0 \cdot m$ by setting $0 \otimes m = \{0, m\}$.

Proof. Denote $a \cdot b = c$, then we have $a \otimes b = \{c, c+m\}$, so we obtain $c \gamma (c+m)$. Add m to $a \otimes b$ thus, $m + a \otimes b = \{c+m, c+2m\}$ so, we have $(c+m) \gamma (c+2m)$. Then, again, we add $2m$ to $a \otimes b$ thus, we have $2m + a \otimes b = \{c+2m, c+3m\}$ so, we have $(c+2m) \gamma (c+3m)$. We continue up to $s-2$, thus we add $(s-2)m$ to $a \otimes b$ and we have $(s-2)m + a \otimes b = \{c+(s-2)m, c+(s-1)m\}$ so, $(c+(s-2)m) \gamma (c+(s-1)m)$. We consider the transitive closure γ^* of the relation γ and we obtain that the elements

$$c, c+m, c+2m, \dots, c+(s-2)m, c+(s-1)m$$

are γ^* equivalent. Denote

$$[c] = \{c, c+m, c+2m, \dots, c+(s-1)m\}.$$

Take any $d \in \mathbf{Z}_n$ and add $d-c$ on $[c]$, then we obtain all classes:

$$d-c+[c] = [d] = \{d, d+m, d+2m, \dots, d+(s-1)m\}$$

For these general classes $[c]$ and $[d]$, we obviously have,

$$[c]+[d] = \{c+d, c+d+m, c+d+2m, \dots, c+d+(s-1)m\} = [c+d]$$

and, as one can see in the next remark, the following is valid:

$$[c] \cdot [d] \subseteq \{c \cdot d, c \cdot d+m, c \cdot d+2m, \dots, c \cdot d+(s-1)m\} = [c \cdot d].$$

Remark 3.6 Let us take $\mathbf{Z}_{16}$ and consider the H_v -field where the only hyperproduct is $2 \otimes 3 = \{6, 14\}$, then we obtain that the fundamental classes are

$$[0] = \{0, 8\}, [1] = \{1, 9\}, [2] = \{2, 10\}, [3] = \{3, 11\},$$

$$[4] = \{4, 12\}, [5] = \{5, 13\}, [6] = \{6, 14\}, [7] = \{7, 15\}.$$

Take the hyperproduct of the special elements, considering them as sets,

$$[2] \cdot [4] = \{2, 10\} \cdot \{4, 12\} = \{8\} \subseteq [8].$$

From this, we have the following remarks. Since the reproduction axiom in the product is not valid, we do not, necessarily, have in the product of classes, consider them as sets, an entire class, thus $[c] \cdot [d] \subseteq [c \cdot d]$, but we consider that the result it is an entire class $[c \cdot d]$. This means that $[c] \otimes [d] = [c \cdot d]$.

On ∂ -hopes we may obtain analogous results: Thus, in integers $(\mathbf{Z}, +)$, fix $n \neq 0$. Take f such that $f(0) = n$ and $f(x) = x$, $\forall x \in \mathbf{Z} - \{0\}$, then $(\mathbf{Z}, \partial) / \beta^* \cong (\mathbf{Z}_n, +)$.

Theorem 3.7 In the integers $(\mathbb{Z}, +, \cdot)$, fix $n \neq 0$, take f such that $f(0)=n$, $f(x)=x$, $\forall x \in \mathbb{Z} - \{0\}$. Then $(\mathbb{Z}, \partial_+, \partial \cdot)$, where ∂_+ , $\partial \cdot$ the ∂ -hopes on sum and product, respectively, is H_v -near-ring, with

$$(\mathbb{Z}, \partial_+, \partial \cdot) / \gamma^* \cong \mathbb{Z}_n.$$

We obtain the same, if we take f such that $f(n)=0$, $f(x)=x$, $\forall x \in \mathbb{Z} - \{n\}$.

Special case is for $n=p$, prime, then $(\mathbb{Z}, \partial_+, \partial \cdot)$ is an H_v -field.

4. HYPER-REPRESENTATIONS

A matrix with entries of an H_v -ring is called H_v -matrix. The product of two H_v -matrices, is defined in the usual manner and it is a set. The n -ary circle hope on the hyper-sum, is used.

The problem of the H_v -matrix representations is the following [5], [10]:

Definition 4.1 For H_v -group $(H, \cdot)$ find H_v -ring $(R, +, \cdot)$, a set $M_R = \{(a_{ij}) \mid a_{ij} \in R\}$ and a map $T: H \rightarrow M_R: h \mapsto T(h)$, then we have a H_v -matrix representation, if

$$T(h_1 h_2) \cap T(h_1) T(h_2) \neq \emptyset, \forall h_1, h_2 \in H.$$

If $T(h_1 h_2) \subset T(h_1) T(h_2)$, T is *inclusion*; if $T(h_1 h_2) = T(h_1) T(h_2) = \{T(h) \mid h \in h_1 h_2\}$, then T is *good*; If T is a good and one to one then it is a *faithful representation*.

Theorem 4.2 In order to have inclusion representation T , of an H_v -group $(H, \cdot)$ by H_v -matrices over $(R, +, \cdot)$, a necessary condition is:

$\forall \beta^*(x)$, $x \in H$ there must exist $a_{ij} \in H$, $i, j \in \{1, \dots, n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, \dots, n\}\}$$

Inclusion representation, induces fundamental homomorphism $T^*: H/\beta^* \rightarrow R/\gamma^*$.

On non-square matrices the hopes *helix*, are defined [4], [5].

The H_v -structures had a variety of applications in many sciences. These range from bio-mathematics, hadronic physics to leptons, to mention but a few. This theory is related to fuzzy theory; consequently, can be widely applicable in industry, too. In books and papers as [1], [3], [5], [6], [7], [10], one can find several applications.

The Lie-Santilli theory on *isotopies* was born in the 1960's to solve Hadronic Mechanics problems, [5], [6], [7], [16], [17]. The *isofields*, needed in this theory correspond to H_v -structures introduced by Santilli & Vougiouklis.

Definition 4.3 $(F, +, \cdot)$, with $(+)$ operation and $(\cdot)$ hope, is *e- H_v -field* if the following are valid: $(F, +)$ is abelian group, $(\cdot)$ is weak associative, $(\cdot)$ is weak distributive to $(+)$, 0 is absorbing, there is scalar unit 1, and $\forall x \in F$ there is a unique inverse x^{-1} : $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

Main e-construction: Given a group $(G, \cdot)$, e unit, define hopes $(\otimes)$ by:

$$x \otimes y = \{xy, g_1, g_2, \dots\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, \dots \in G - \{e\}$$

$(G, \otimes)$ is H_b -group containing $(G, \cdot)$. $(G, \otimes)$ is *e- H_v -group*.

Example 4.4 In quaternions $Q = \{1, -1, i, -i, j, -j, k, -k\}$, $i^2 = j^2 = -1$, $ij = -ji = k$, denote $\underline{i} = \{i, -i\}$, $\underline{j} = \{j, -j\}$, $\underline{k} = \{k, -k\}$. We can define several hopes $(*)$ by enlarging only few products remaining in the same classes. For example, we can take some results like $(-1)*\underline{k} = \underline{k}$, $k*i = \underline{j}$, $i*j = \underline{k}$. Then $(Q, *)$ is *e- H_v -group*.

5. ON 2×2 RAISED H_v -MATRIX REPRESENTATIONS

We focus on *small non-degenerate H_v -fields* on $(Z_n, +, \cdot)$, which, in isothory, fulfil: (1) *very-thin minimal*, (2) *weak commutative*, (3) *have 0, 1, scalars*, (4) *all inverses are unique*.

Therefore, we cannot raise the result if it is 1 and we cannot put 1 in enlargement.

We present some known results and new examples on the topic [20], [21], [23].

The multiplicative H_v -fields on $(Z_4, +, \cdot)$, with non-degenerate fundamental field, satisfying the above 4 conditions, are isomorphic to ones, where the only product is set, is $2 \otimes 3 = \{0, 2\}$ or $3 \otimes 2 = \{0, 2\}$. The fundamental classes are: $[0] = \{0, 2\}$, $[1] = \{1, 3\}$ and $(Z_4, +, \otimes) / \gamma^* \cong (Z_2, +, \cdot)$.

Example 5.1 Take the 2×2 upper triangular H_v -matrices on the above H_v -field $(Z_4, +, \otimes)$ of the case that only $2 \otimes 3 = \{0, 2\}$ is a hyperproduct. Consider the set

$$\begin{pmatrix} \{1, 3\} & \{0, 1, 2, 3\} \\ 0 & \{1, 3\} \end{pmatrix}$$

So, we have the 16 elements:

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, a_3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, a_4 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, a_5 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, a_6 = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, a_7 = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, a_8 = \begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix}$$

$$a_9 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, a_{10} = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, a_{11} = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}, a_{12} = \begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix}, a_{13} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, a_{14} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}, a_{15} = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}, a_{16} = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}.$$

Then, we obtain the following

$\otimes$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
a_1	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
a_2	a_2	a_3	a_4	a_1	a_8	a_5	a_6	a_7	a_{10}	a_{11}	a_{12}	a_9	a_{16}	a_{13}	a_{14}	a_{15}
a_3	a_3	a_4	a_1	a_2	a_5, a_7	a_6, a_8	a_5, a_7	a_6, a_8	a_{11}	a_{12}	a_9	a_{10}	a_{13}, a_{15}	a_{14}, a_{16}	a_{13}, a_{15}	a_{14}, a_{16}
a_4	a_4	a_1	a_2	a_3	a_6	a_7	a_8	a_5	a_{12}	a_9	a_{10}	a_{11}	a_{14}	a_{15}	a_{16}	a_{13}
a_5	a_5	a_6	a_7	a_8	a_1	a_2	a_3	a_4	a_{13}	a_{14}	a_{15}	a_{16}	a_9	a_{10}	a_{11}	a_{12}
a_6	a_6	a_7	a_8	a_5	a_4	a_1	a_2	a_3	a_{14}	a_{15}	a_{16}	a_{13}	a_{12}	a_9	a_{10}	a_{11}
a_7	a_7	a_8	a_5	a_6	a_1, a_3	a_2, a_4	a_1, a_3	a_2, a_4	a_{15}	a_{16}	a_{13}	a_{14}	a_9, a_{11}	a_{10}, a_{12}	a_9, a_{11}	a_{10}, a_{12}
a_8	a_8	a_5	a_6	a_7	a_2	a_3	a_4	a_1	a_{16}	a_{13}	a_{14}	a_{15}	a_{10}	a_{11}	a_{12}	a_9
a_9	a_9	a_{12}	a_{11}	a_{10}	a_{13}	a_{16}	a_{15}	a_{14}	a_1	a_4	a_3	a_2	a_5	a_8	a_7	a_6
a_{10}	a_{10}	a_9	a_{12}	a_{11}	a_{16}	a_{15}	a_{14}	a_{13}	a_2	a_1	a_4	a_3	a_8	a_7	a_6	a_5
a_{11}	a_{11}	a_{10}	a_9	a_{12}	a_{13}, a_{15}	a_{14}, a_{16}	a_{13}, a_{15}	a_{14}, a_{16}	a_3	a_2	a_1	a_4	a_5, a_7	a_6, a_8	a_5, a_7	a_6, a_8
a_{12}	a_{12}	a_{11}	a_{10}	a_9	a_{14}	a_{15}	a_{16}	a_{13}	a_4	a_3	a_2	a_1	a_6	a_5	a_8	a_7
a_{13}	a_{13}	a_{16}	a_{15}	a_{14}	a_9	a_{12}	a_{11}	a_{10}	a_5	a_8	a_7	a_6	a_1	a_4	a_3	a_2
a_{14}	a_{14}	a_{13}	a_{16}	a_{15}	a_{12}	a_{11}	a_{10}	a_9	a_6	a_5	a_8	a_7	a_4	a_3	a_2	a_1
a_{15}	a_{15}	a_{14}	a_{13}	a_{16}	a_9, a_{11}	a_{10}, a_{12}	a_9, a_{11}	a_{10}, a_{12}	a_7	a_6	a_5	a_8	a_{14}, a_{13}	a_{12}, a_{14}	a_{11}, a_{13}	a_{12}, a_{14}
a_{16}	a_{16}	a_{15}	a_{14}	a_{13}	a_{10}	a_9	a_{12}	a_{11}	a_8	a_7	a_6	a_5	a_{12}	a_{11}	a_{14}	a_{13}

Table 1

Remark that in the above H_V -structure, 32 of 256 results are multivalued with cardinality 2. It is left and right reproductive, so it is an H_V -group.

Let us consider the multiplicative H_V -fields on $(\mathbb{Z}_6, +, \cdot)$, with non-degenerate fundamental field, satisfying the above 4 conditions, where the only product is set, is $3 \otimes 5 = \{0, 3\}$ or $4 \otimes 2 = \{2, 5\}$. Fundamental classes: $[0] = \{0, 3\}$, $[1] = \{1, 4\}$, $[2] = \{2, 5\}$ and $(\mathbb{Z}_6, +, \otimes) / \gamma^* \cong (\mathbb{Z}_3, +, \cdot)$.

Example 5.2 Take the 2×2 upper triangular H_V -matrices on the above H_V -field $(\mathbb{Z}_6, +, \otimes)$ of the case that only $3 \otimes 5 = \{0, 3\}$ is a hyperproduct. Consider the set

$$\begin{pmatrix} \{1, 3\} & \mathbb{Z}_6 \\ 0 & 1 \end{pmatrix}$$

So, we have the 12 elements:

$$b_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, b_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b_3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, b_4 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, b_5 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, b_6 = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix},$$

$$b_7 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, b_8 = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, b_9 = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}, b_{10} = \begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix}, b_{11} = \begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}, b_{12} = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}.$$

Then, we obtain the following

$\otimes$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}
b_1	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}
b_2	b_2	b_3	b_4	b_5	b_6	b_1	b_8	b_9	b_{10}	b_{11}	b_{12}	b_7
b_3	b_3	b_4	b_5	b_6	b_1	b_2	b_9	b_{10}	b_{11}	b_{12}	b_7	b_8
b_4	b_4	b_5	b_6	b_1	b_2	b_3	b_{10}	b_{11}	b_{12}	b_7	b_8	b_9
b_5	b_5	b_6	b_1	b_2	b_3	b_4	b_{11}	b_{12}	b_7	b_8	b_9	b_{10}
b_6	b_6	b_1	b_2	b_3	b_4	b_5	b_{12}	b_7	b_8	b_9	b_{10}	b_{11}
b_7	b_7	b_{10}	b_7	b_{10}	b_7	b_7, b_{10}	b_7	b_{10}	b_7	b_{10}	b_7	b_7, b_{10}
b_8	b_8	b_{11}	b_8	b_{11}	b_8	b_8, b_{11}	b_8	b_{11}	b_8	b_{11}	b_8	b_8, b_{11}
b_9	b_9	b_{12}	b_9	b_{12}	b_9	b_9, b_{12}	b_9	b_{12}	b_9	b_{12}	b_9	b_9, b_{12}
b_{10}	b_{10}	b_7	b_{10}	b_7	b_{10}	b_7, b_{10}	b_{10}	b_7	b_{10}	b_7	b_{10}	b_7, b_{10}
b_{11}	b_{11}	b_8	b_{11}	b_8	b_{11}	b_8, b_{11}	b_{11}	b_8	b_{11}	b_8	b_{11}	b_8, b_{11}
b_{12}	b_{12}	b_9	b_{12}	b_9	b_{12}	b_9, b_{12}	b_{12}	b_9	b_{12}	b_9	b_{12}	b_9, b_{12}

Table 2

In the above H_V -structure, 12 of 144 results are multivalued of cardinality 2. However, it is not left or right reproductive, so it remains an H_V -semigroup.

Example 5.3 Take the 2×2 upper triangular H_V -matrices on the H_V -field $(Z_6, +, \otimes)$ of the case that only $4 \otimes 2 = \{2, 5\}$ is a hyperproduct. Consider the set

$$\begin{pmatrix} \{1, 4\} & Z_6 \\ 0 & 1 \end{pmatrix}.$$

So, we have the 12 elements:

$$b_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, b_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b_3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, b_4 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, b_5 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, b_6 = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix},$$

$$b_7 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, b_8 = \begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix}, b_9 = \begin{pmatrix} 4 & 2 \\ 0 & 1 \end{pmatrix}, b_{10} = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}, b_{11} = \begin{pmatrix} 4 & 4 \\ 0 & 1 \end{pmatrix}, b_{12} = \begin{pmatrix} 4 & 5 \\ 0 & 1 \end{pmatrix},$$

Then, we obtain the following

$\otimes$	b₁	b₂	b₃	b₄	b₅	b₆	b₇	b₈	b₉	b₁₀	b₁₁	b₁₂
b₁	b ₁	b ₂	b ₃	b ₄	b ₅	b ₆	b ₇	b ₈	b ₉	b ₁₀	b ₁₁	b ₁₂
b₂	b ₂	b ₃	b ₄	b ₅	b ₆	b ₁	b ₈	b ₉	b ₁₀	b ₁₁	b ₁₂	b ₇
b₃	b ₃	b ₄	b ₅	b ₆	b ₁	b ₂	b ₉	b ₁₀	b ₁₁	b ₁₂	b ₇	b ₈
b₄	b ₄	b ₅	b ₆	b ₁	b ₂	b ₃	b ₁₀	b ₁₁	b ₁₂	b ₇	b ₈	b ₉
b₅	b ₅	b ₆	b ₁	b ₂	b ₃	b ₄	b ₁₁	b ₁₂	b ₇	b ₈	b ₉	b ₁₀
b₆	b ₆	b ₁	b ₂	b ₃	b ₄	b ₅	b ₁₂	b ₇	b ₈	b ₉	b ₁₀	b ₁₁
b₇	b ₇	b ₁₁	b ₉ ,b ₁₂	b ₇	b ₁₁	b ₉	b ₇	b ₁₁	b ₉ ,b ₁₂	b ₇	b ₁₁	b ₉
b₈	b ₈	b ₁₂	b ₇ ,b ₁₀	b ₈	b ₁₂	b ₁₀	b ₈	b ₁₂	b ₇ ,b ₁₀	b ₈	b ₁₂	b ₁₀
b₉	b ₉	b ₇	b ₈ ,b ₁₁	b ₉	b ₇	b ₁₁	b ₉	b ₇	b ₈ ,b ₁₁	b ₉	b ₇	b ₁₁
b₁₀	b ₁₀	b ₈	b ₉ ,b ₁₂	b ₁₀	b ₈	b ₁₂	b ₁₀	b ₈	b ₉ ,b ₁₂	b ₁₀	b ₈	b ₁₂
b₁₁	b ₁₁	b ₉	b ₇ ,b ₁₀	b ₁₀	b ₉	b ₇	b ₁₁	b ₉	b ₇ ,b ₁₀	b ₁₁	b ₉	b ₇
b₁₂	b ₁₂	b ₁₃	b ₈ ,b ₁₁	b ₁₂	b ₁₀	b ₈	b ₁₂	b ₁₀	b ₈ ,b ₁₁	b ₁₂	b ₁₀	b ₈

Table 3

In the above H_v -structure, 12 of 144 results are multivalued of cardinality 2. It is not left or right reproductive, so it is an H_v -semigroup.

Let us consider the multiplicative H_v -field on $(\mathbb{Z}_9, +, \cdot)$, satisfying the above 4 conditions, where the only product is set, is $2 \otimes 8 = \{4, 7\}$. Fundamental classes: $[0] = \{0, 3, 6\}$, $[1] = \{1, 4, 7\}$, $[2] = \{2, 5, 8\}$ and $(\mathbb{Z}_9, +, \otimes) / \gamma^* \cong (\mathbb{Z}_3, +, \cdot)$.

Example 5.4 Take the 2×2 upper triangular H_v -matrices on the H_v -field $(\mathbb{Z}_9, +, \otimes)$ of the case that only $2 \otimes 8 = \{4, 7\}$ is a hyperproduct. Consider the set

$$\begin{pmatrix} 1 & \mathbb{Z}_9 \\ 0 & \{1, 8\} \end{pmatrix}.$$

So, we have the 18 elements:

$$c_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, c_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, c_3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, c_4 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, c_5 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, c_6 = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix},$$

$$c_7 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, c_8 = \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}, c_9 = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}, c_{10} = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}, c_{11} = \begin{pmatrix} 1 & 1 \\ 0 & 8 \end{pmatrix}, c_{12} = \begin{pmatrix} 1 & 2 \\ 0 & 8 \end{pmatrix},$$

$$c_{13} = \begin{pmatrix} 1 & 3 \\ 0 & 8 \end{pmatrix}, c_{14} = \begin{pmatrix} 1 & 4 \\ 0 & 8 \end{pmatrix}, c_{15} = \begin{pmatrix} 1 & 5 \\ 0 & 8 \end{pmatrix}, c_{16} = \begin{pmatrix} 1 & 6 \\ 0 & 8 \end{pmatrix}, c_{17} = \begin{pmatrix} 1 & 7 \\ 0 & 8 \end{pmatrix}, c_{18} = \begin{pmatrix} 1 & 8 \\ 0 & 8 \end{pmatrix}.$$

Then, we obtain the following

$\otimes$	c1	c2	c3	c4	c5	c6	c7	c8	c9	c10	c11	c12	c13	c14	c15	c16	c17	c18
c1	c1	c2	c3	c4	c5	c6	c7	c8	c9	c10	c11	c12	c13	c14	c15	c16	c17	c18
c2	c2	c3	c4	c5	c6	c7	c8	c9	c1	c18	c10	c11	c12	c13	c14	c15	c16	c17
c3	c3	c4	c5	c6	c7	c8	c9	c1	c2	c14,c17	c15,c18	c10,c16	c11,c17	c13,c18	c10,c13	c13,c12	c12,c15	c13,c16
c4	c4	c5	c6	c7	c8	c9	c1	c2	c3	c16	c17	c18	c10	c11	c12	c13	c14	c15
c5	c5	c6	c7	c8	c9	c1	c2	c3	c4	c15	c16	c17	c18	c10	c11	c12	c13	c14
c6	c6	c7	c8	c9	c1	c2	c3	c4	c5	c14	c15	c16	c17	c18	c10	c11	c12	c13
c7	c7	c8	c9	c1	c2	c3	c4	c5	c6	c13	c14	c15	c16	c17	c18	c10	c11	c12
c8	c8	c9	c1	c2	c3	c4	c5	c6	c7	c12	c13	c14	c15	c16	c17	c18	c10	c11
c9	c9	c1	c2	c3	c4	c5	c6	c7	c8	c11	c12	c13	c14	c15	c16	c17	c18	c10
c10	c10	c11	c12	c13	c14	c15	c16	c17	c18	c1	c2	c3	c4	c5	c6	c7	c8	c9
c11	c11	c12	c13	c14	c15	c16	c17	c18	c9	c1	c2	c3	c4	c5	c6	c7	c8	
c12	c12	c13	c14	c15	c16	c17	c18	c10	c11	c5,c8	c6,c9	c1,c7	c2,c8	c3,c9	c1,c4	c2,c5	c3,c6	c4,c7
c13	c13	c14	c15	c16	c17	c18	c10	c11	c12	c7	c8	c9	c1	c2	c3	c4	c5	c6
c14	c14	c15	c16	c17	c18	c10	c11	c12	c13	c6	c7	c8	c9	c1	c2	c3	c4	c5
c15	c15	c16	c17	c18	c10	c11	c12	c13	c14	c5	c6	c7	c8	c9	c1	c2	c3	c4
c16	c16	c17	c18	c10	c11	c12	c13	c14	c15	c4	c5	c6	c7	c8	c9	c1	c2	c3
c17	c17	c18	c10	c11	c12	c13	c14	c15	c16	c3	c4	c5	c6	c7	c8	c9	c1	c2
c18	c18	c10	c11	c12	c13	c14	c15	c16	c17	c2	c3	c4	c5	c6	c7	c8	c9	c1

Table 4

In this H_V -structure, only 18 of 324 results are multivalued of cardinality 2. It is left and right reproductive, therefore, it is an H_V -group.

Take the multiplicative H_V -fields on $(\mathbb{Z}_{10}, +, \cdot)$, satisfying the above 4 conditions, where the only product is set is $3 \otimes 6 = \{3, 8\}$ or $2 \otimes 6 = \{2, 7\}$. The fundamental classes are: $[0] = \{0, 5\}$, $[1] = \{1, 6\}$, $[2] = \{2, 7\}$, $[3] = \{3, 8\}$, $[4] = \{4, 9\}$ and $(\mathbb{Z}_{10}, +, \otimes) / \gamma^* \cong (\mathbb{Z}_5, +, \cdot)$.

Example 5.5 Take the 2×2 upper triangular H_V -matrices on the H_V -field $(\mathbb{Z}_{10}, +, \otimes)$ of the case that only $3 \otimes 6 = \{3, 8\}$ is a hyperproduct. Consider the set

$$\left\{ \begin{pmatrix} 1 & \mathbb{Z}_{10} \\ 0 & 6 \end{pmatrix} \right\}.$$

So, we have the 10 elements:

$$\begin{aligned} d_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}, d_2 = \begin{pmatrix} 1 & 1 \\ 0 & 6 \end{pmatrix}, d_3 = \begin{pmatrix} 1 & 2 \\ 0 & 6 \end{pmatrix}, d_4 = \begin{pmatrix} 1 & 3 \\ 0 & 6 \end{pmatrix}, d_5 = \begin{pmatrix} 1 & 4 \\ 0 & 6 \end{pmatrix}, \\ d_6 &= \begin{pmatrix} 1 & 5 \\ 0 & 6 \end{pmatrix}, d_7 = \begin{pmatrix} 1 & 6 \\ 0 & 6 \end{pmatrix}, d_8 = \begin{pmatrix} 1 & 7 \\ 0 & 6 \end{pmatrix}, d_9 = \begin{pmatrix} 1 & 8 \\ 0 & 6 \end{pmatrix}, d_{10} = \begin{pmatrix} 1 & 9 \\ 0 & 6 \end{pmatrix}. \end{aligned}$$

Then, we obtain the following

$\otimes$	d1	d2	d3	d4	d5	d6	d7	d8	d9	d10
d1	d1	d2	d3	d4	d5	d6	d7	d8	d9	d10
d2	d7	d8	d9	d10	d1	d2	d3	d4	d5	d6
d3	d3	d4	d5	d6	d7	d8	d9	d10	d1	d2
d4	d4,d9	d5,d10	d1,d6	d2,d7	d3,d8	d4,d9	d5,d10	d1,d6	d2,d7	d3,d8
d5	d5	d6	d7	d8	d9	d10	d1	d2	d3	d4
d6	d1	d2	d3	d4	d5	d6	d7	d8	d9	d10
d7	d7	d8	d9	d10	d1	d2	d3	d4	d5	d6
d8	d3	d4	d5	d6	d7	d8	d9	d10	d1	d2
d9	d9	d10	d1	d2	d3	d4	d5	d6	d7	d8
d10	d5	d6	d7	d8	d9	d10	d1	d2	d3	d4

Table 5

In this H_V -structure, 10 of 100 results are multivalued of cardinality 2, where only the matrix d_4 on the left, gives multivalued results. It is left and right reproductive, so, it is an H_V -group.

Example 5.6 Take the 2×2 upper triangular H_V -matrices on the H_V -field $(\mathbb{Z}_{10}, +, \otimes)$ of the case that only $2 \otimes 6 = \{2, 7\}$ is a hyperproduct. Consider the set

$$\left\{ \begin{pmatrix} 1 & \mathbb{Z}_{10} \\ 0 & \{1, 6\} \end{pmatrix} \right\}$$

So, we have the 20 elements:

$$d_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, d_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, d_3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, d_4 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, d_5 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, d_6 = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}, d_7 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix},$$

$$d_8 = \begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}, d_9 = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}, d_{10} = \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}, d_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}, d_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 6 \end{pmatrix}, d_{13} = \begin{pmatrix} 1 & 2 \\ 0 & 6 \end{pmatrix}, d_{14} = \begin{pmatrix} 1 & 3 \\ 0 & 6 \end{pmatrix},$$

$$d_{15} = \begin{pmatrix} 1 & 4 \\ 0 & 6 \end{pmatrix}, d_{16} = \begin{pmatrix} 1 & 5 \\ 0 & 6 \end{pmatrix}, d_{17} = \begin{pmatrix} 1 & 6 \\ 0 & 6 \end{pmatrix}, d_{18} = \begin{pmatrix} 1 & 7 \\ 0 & 6 \end{pmatrix}, d_{19} = \begin{pmatrix} 1 & 8 \\ 0 & 6 \end{pmatrix}, d_{20} = \begin{pmatrix} 1 & 9 \\ 0 & 6 \end{pmatrix}.$$

Then, we obtain the following

$\otimes$	d1	d2	d3	d4	d5	d6	d7	d8	d9	d10	d11	d12	d13	d14	d15	d16	d17	d18	d19	d20
d1	d1	d2	d3	d4	d5	d6	d7	d8	d9	d10	d11	d12	d13	d14	d15	d16	d17	d18	d19	d20
d2	d2	d1	d3	d4	d5	d6	d7	d8	d9	d10	d11	d12	d13	d14	d15	d16	d17	d18	d19	d20
d3	d3	d1	d1	d5	d6	d7	d8	d9	d10	d11	d12	d13	d14	d15	d16	d17	d18	d19	d20	d20
d4	d4	d1	d1	d5	d6	d7	d8	d9	d10	d11	d12	d13	d14	d15	d16	d17	d18	d19	d20	d20
d5	d5	d6	d7	d8	d1	d10	d11	d12	d13	d14	d15	d16	d17	d18	d19	d20	d20	d20	d20	d20
d6	d6	d7	d8	d1	d10	d11	d12	d13	d14	d15	d16	d17	d18	d19	d20	d20	d20	d20	d20	d20
d7	d7	d8	d1	d10	d11	d12	d13	d14	d15	d16	d17	d18	d19	d20	d20	d20	d20	d20	d20	d20
d8	d8	d1	d10	d11	d12	d13	d14	d15	d16	d17	d18	d19	d20	d20	d20	d20	d20	d20	d20	d20
d9	d9	d10	d11	d12	d13	d14	d15	d16	d17	d18	d19	d20	d20	d20	d20	d20	d20	d20	d20	d20
d10	d10	d11	d12	d13	d14	d15	d16	d17	d18	d19	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20
d11	d11	d12	d13	d14	d15	d16	d17	d18	d19	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20
d12	d12	d13	d14	d15	d16	d17	d18	d19	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20
d13	d13	d14	d15	d16	d17	d18	d19	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20
d14	d14	d15	d16	d17	d18	d19	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20
d15	d15	d16	d17	d18	d19	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20
d16	d16	d17	d18	d19	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20
d17	d17	d18	d19	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20
d18	d18	d19	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20
d19	d19	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20
d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20	d20

Table 6

In this H_V -structure, only 20 of 400 results are multivalued of cardinality 2. It is not left or right reproductive, so, it remains an H_V -semigroup.

6. CONCLUSIONS

The class of H_V -structures is extremely large, therefore, the H_V -structures can be used widely as mathematical models. In the case of representations of Lie-Santilli's admissibility appropriate H_V -fields are need which have special properties. The raised H_V -fields are the smallest hyperstructures which have all needed properties. Moreover, these H_V -fields give interesting representing sets of hypermatrices as new hyperstructures.

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**SOME DEVELOPMENTS AND APPLICATIONS ON FIELD
THEORY IN GEOMETRY**

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Abstract

The dimension in geometry may be extended to n -dimension. The field may be developed from scalar and vector to tensor, spinor, torsion and twistor, etc. Further, they are applied to physics, biology, earthquakes and social science, etc. Field theory has been widely applied in many regions of natural and social sciences, and its any development will necessarily inspire and apply to more aspects.

Keywords: geometry, field theory, dimension, tensor, spinor, torsion, twistor, physics, biology, social science.

I. Introduction

It is known that the space-dimension of mathematics and calculus are extended to higher n-dimensions geometry [1]. It may be Hilbert space. In mathematics, physics and many regions the field theory in geometry is all a very important problem. When the space-dimension of calculus is extended to n, fractal and complex, and various number-systems, the field theory and its formulas may be correspondingly extended. In these cases, Gauss's theorem and Stokes's theorem, and corresponding extensions on gradient, divergence and curl are searched [2].

From 1978 Santilli discovered the new Iso-Mathematics, which includes isonumbers, isorepresentation [3-5], etc. In Iso-Mathematics, isonumbers and genonumbers of dimensions 1, 2, 4, 8, and their isoduals and pseudoduals may extend to "hidden numbers" of dimension 3, 5, 6, 7 [4]. They are applied for many regions [5,6].

A known important representation is tree in the graph theory. We extended tree, and proposed a new tree-field representation (Fig.1), which includes two parts: tree (trunk) and field [7].

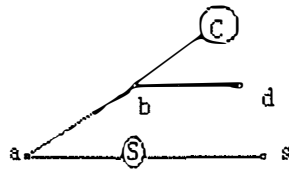


Fig.1. Tree-field representation

Here the fields (C and S, etc.) are some sets of much small trees, which are also trees under microscope. For a certainty condition the few small trees in the field can become a usual tree as trunk. These may be the directed graph or undirected graph. In Fig.1 the vertexes b, d, s and the fields C, S possess the same macroscopic property, and fields are only minute detail on vertex. In the tree-field representation there is a relation (for vertex n, edge m and field s) $m = n + s - 1$. Further, the field may have different weights and levels, and may be extended to the graph theory, i.e., $G=(V,E,F)$, in which F is field and a set of the small graph, and represents region, grove and forests, etc.

For two any sets M and N, we researched the simple mathematical joint relation of algebra in set theory

$$(M, k)(k', N) = (M, kk', N), \quad (1)$$

and discussed its some properties. It may describe various relations among sciences and many intersecting sciences. This is applied to various aspects in natural science and social science, which includes transcription, splicing and replication, etc., in biology, and both main relations in micro-economics and

macro-economics, etc. Further, it may be corrected and developed [8]. In this paper, we discuss some developments and applications of field theory.

2. Various Tensor and Spinor Fields

The field in mathematics may be extended from usual scalar field, vector field to tensor field in general relativity, and spinor field of particles, torsion and twistor fields, etc.

A classical vector field is the electromagnetic field. In electrodynamics Maxwell equations are [9]:

$$\frac{\partial F_{ik}}{\partial x_k} = \frac{4\pi}{c} j_i \text{ (flow vector)}. \quad (2)$$

In general relativity the gravitational field is tensor, in which there is Einstein symmetric tensor:

$$G_{\mu\nu} = R_{\mu\nu} - (Rg_{\mu\nu}/2). \quad (3)$$

$R_{ik,l}$ is 3-order tensor, etc. And the divergence of $G_{\mu\nu}$ is zero. In Riemann geometry an extensive form of curl is Christoffel symbols [9]:

$$\Gamma_{kl}^i = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right). \quad (4)$$

It is not tensor. But, there have Riemann curvature tensor, Ricci tensor, scalar curvature and Contorsion tensor, etc.

Further, it may be extended to Brans-Dicke scalar-tensor theory [10], gravitational vector-tensor theory and tensor-tensor theory, both may be composed of the gravitational field and electromagnetic field and the electromagnetic general relativity [11], etc.

The gravitational field and electromagnetic field try to be unified by the gauge invariance geometry, the five-dimensional Kaluza-Klein theory, the projective theory, the affine field and the bivector fields, the non-symmetric field [12,13], and Utiyama field [14], Yilmaz theory [15] and Carmili theory [16], etc.

We proposed that the gravitational field and the source-free electromagnetic field can be unified preliminarily by the equations in the Riemannian geometry [17,18]:

$$R_{klm}^i = \kappa T_{klm}^i. \quad (5)$$

Both fields are contractions of im and ik, respectively. If $R_{klm}^i = \kappa T_{klm}^i = \text{constant}$, so it will be equivalent to the Yang's gravitational equations [19]:

$$R_{km;l} - R_{kl;m} = 0, \quad (6)$$

which include $R_{lm} = 0$. From $R_{lm} = 0$ we can obtain the Lorentz equations of motion, the first system and second source-free system of Maxwell field

equations. This unification can be included in the gauge theory, and the unified gauge group is $SL(2,C) \times U(1) = GL(2,C)$, which is just the same as the gauge group of the Riemannian manifold. Another unification on the general nonsymmetric metric field with high-dimensional space-time is analyzed mathematically. Moreover, some possible unification ways on the gravitational field and electromagnetic field are discussed [18].

The physical 3-dimensional space has been developed to 4-dimensional space-time of relativity, and Kaluza-Klein's 5-dimensional space, and higher dimensional space of superstring, etc. Further, higher-dimensional space is related to black hole [20-22] and to Ruppeiner geometry and Weinhold geometry [23]. The physical space-time developed to fractal and complex dimension derive necessarily the fractal relativity [24,25] and fractal and complex dimensional time [26,27,7], etc.

The unification of various interactions is always an important question in physics. Their mathematical basis is mainly the gauge groups. Weinberg and Salam proposed a well-known electroweak theory unified the weak and electromagnetic interactions, whose unified gauge group is $SU(2) \otimes U(1) = U(2)$ [28,29]. Further, various grand unified theories of the strong, weak and electromagnetic interactions are researched, whose pioneer is Bars-Halpern-Yoshimura model [30,31].

Einstein gravitational Lagrangian possesses two invariances: the $GL(4,R)$ invariance of Einstein under coordinate transformations, and the $SL(2,C)$ gauge invariant of Weyl. The strong interaction of quarks possesses internal $SU(3)$ symmetry. From these symmetries Isham, Salam and Strathdee proposed the unified scheme on gravitational and strong interactions, whose gauge group is $SU(3) \otimes SL(2,C) = SL(6,C)$ [32-34]. Based on the Weinberg-Salam unified electroweak theory and Isham-Salam-Strathdee strong-gravitational interactions unified scheme [32-34], in 1974 we proposed that the simplest unified gauge group of four-interactions must be $GL(6,C)$ [17,35], which is related to four- or six-dimensional Riemann geometry and nonsymmetric field [12,35]. In 1977 Terazawa proposed also a unified gauge symmetry of $GL(32N,C)$ or $GL(12+2n,C)$ for combining the $SU(16N)$ or $SU(6+n)$ group of lepton-quark internal symmetry and the $SL(2,C)$ Lorentz group of space-time symmetry (where $N=1,2,3,\dots$ and $n=0,1,2,3,\dots,N$) [36]. Of course, this unified group is a non-compact group, and seems to construct a 'no-go' law. But, it is not only reasonable but also necessary, because the classical gravitational interaction corresponds to a non-compact group $SL(2,C)$. Further, it should be that the gravitational theory is developed to the quantum gravity theory, for example, the supergravity and the loop quantum theory [37-39], etc.

Based on Some new representations of the supersymmetric transformations and the supermultiplet, Graded Lie Algebras and various formulations (equations, commutation relations, propagators, Jacobi identities,

etc.) of bosons and fermions may be unified. On the one hand, the mathematical characteristic of particles is proposed: bosons correspond to real number, and fermions correspond to imaginary number, respectively. Such fermions of even (or odd) number form bosons (or fermions), which is just consistent with a relation between imaginary and real number. The imaginary number is only included in the equations, forms, and matrixes of fermions. It is connected with relativity. On the other hand, the unified forms of supersymmetry are also connected with the statistics unifying Bose-Einstein and Fermi-Dirac statistics, and with the possible violation of Pauli exclusion principle; and a unified partition function is obtained. Moreover, three quarks may be described by the Borromean rings. The quantum statistics is unified by the nonlinear equations. A developed direction of particle physics and modern science is possibly the higher dimensional complex space [35].

Moreover, the unification is connected with the string model [40,41] and the grand unification theory [41], and the supersymmetry [42] and supergravity, etc.

It is believed that all the Lorentz covariant equations are the tensor equations. But, then it has been found that spins s of many particles are not zero, and in particular, the particle of spin $s=1/2$ is introduced into the two-component spin-field equation, which include Weyl equation, Dirac equations and Dirac-Fierz-Pauli equations with any spin, whose forms of spinor are [43] are:

$$\partial_{ab} \phi^{aa_1 a_2 \dots a_l}_{b_1 b_2 \dots b_k} = im \chi^{a_1 a_2 \dots a_l}_{bb_1 b_2 \dots b_k}, \quad \partial^{ab} \chi^{a_1 a_2 \dots a_l}_{bb_1 b_2 \dots b_k} = im \phi^{aa_1 a_2 \dots a_l}_{b_1 b_2 \dots b_k}. \quad (7)$$

In 1932 van der Waerden proposed spinor analysis, whose two-component spinor differential operator is:

$$\partial_{11} = \partial_3 - i\partial_4, \quad \partial_{12} = \partial_1 - i\partial_2, \quad \partial_{21} = \partial_1 + i\partial_2, \quad \partial_{22} = -\partial_3 - i\partial_4. \quad (8)$$

At present, it is considered that the most common Lorentz covariant equation must be spinor equation [43]. It is clear that formula (8) is similar to curl.

Horvathy, et al., proposed the bosonized supersymmetry and massive higher-spin fields from (3+1)-dimensional linear vector equations and the Majorana-Dirac-Staunton theory [44]. Smilga discussed physics of crypto-Hermitian and crypto-supersymmetric field theories [45]. DeWolfe, et al., discussed gravity dual of metastable dynamical supersymmetry breaking [46]. Aguirre, et al., searched fractal structures of nonlinear dynamics in various different fields [47].

3. Various Torsion and Twistor Fields

Kleinert studies the multivalued fields in condensed matter, electromagnetism and gravitation [48]. In general metric-affine geometry

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2}(\Gamma_{\mu\nu}^{\lambda} + \Gamma_{\nu\mu}^{\lambda}) + \frac{1}{2}(\Gamma_{\mu\nu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda}), \quad (9)$$

which divides into symmetrical part and antisymmetric part, and both are 40 and 24 degrees of freedom, respectively. Here antisymmetric part is called torsion [48]:

$$T_{\mu\nu}^{\lambda} = \frac{1}{2}(\Gamma_{\mu\nu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda}). \quad (10)$$

Spin tensor corresponds to torsion [49].

General curved space has curvature and torsion. General relativity has curvature, and has not torsion ($T=0$). When Riemann-Cartan space-time is simplified to Weitzenbock space-time, it is 16 degrees of freedom, and is torsion and is not curvature ($R=0$) [50]. Ordinary flat space is not curvature or torsion. Torsion field is possibly the theoretical foundation of quantum entanglement.

Twistor is proposed by Penrose [51]. It is a new type of algebra by description for Minkowski space-time, in terms of which it is possible to express any conformally covariant or Poincaré covariant operation. The elements of the algebra (twistors) are combined according to tensor-type rules, but they differ from tensors or spinors in that they describe locational properties in addition to directional ones. The representation of a null line by a pair of two-component spinors, one of which defines the direction of the line and the other, its moment about the origin, gives the simplest type of twistor, with four complex components.

Penrose, et al., proposed twistor theory as an approach to the quantisation of fields and space-time [52]. Penrose searched nonlinear gravitons and curved twistor theory. Twistor space is a parameter space-time with complex structure [53]. Twistor is:

$$Z_{\alpha} = (\lambda_A, \bar{\mu}^{\dot{A}}), \quad \bar{Z}^{\alpha} = (\mu^A, \bar{\lambda}_{\dot{A}}). \quad (11)$$

Here a pair of two-component spinors is:

$$Z_{;1}^A = (\omega^{\alpha}, \bar{\pi}_{\dot{\beta}}), \quad Z_{;2}^A = (\lambda^{\alpha}, \bar{\eta}_{\dot{\beta}}). \quad (12)$$

A 4-dimensional point is represented by 3-dimensional complex space, i.e., complex number in twistor. It is applied to quantization and curved space-time.

Penrose and Rindler discussed spinor and twistor methods in space-time geometry [54]. Twistor theory offers a new approach to the synthesis of quantum theory and relativity. Twistors for flat space-time are the $SU(2,2)$ spinors of the twofold covering group $O(2,4)$ of the conformal group. Penrose's basic relations are [54]:

$$\omega^{\alpha} = iz^{\alpha\dot{\beta}} \bar{\pi}_{\dot{\beta}}, \quad \bar{\omega}^{\dot{\alpha}} = -i\pi_{\beta} \bar{z}^{\beta\dot{\alpha}}. \quad (13)$$

They describe the momentum and angular momentum structure of zero-rest-mass particles. Space-time points arise as secondary concepts corresponding to linear sets in twistor space. Twistors are represented in two-component spinor terms. The generalisation to curved space can be accomplished in three ways; i) local twistors, a conformally invariant calculus, ii) global twistors, and iii) asymptotic twistors which provide the framework for an S-matrix approach in asymptotically flat space-times.

Further, Hayashi discussed general relativity as gauge field theory in curved twistor space [55]. Penrose summarized the central programme of twistor theory, which includes twistor quantization, self-dual and anti-self-dual fields, helicity 3/2 fields and the vacuum equations, etc [56]. Twistor may be related to string, superstring and supersymmetry [57-59], twistor space [60,61], twistor transform in d dimensions and a unifying role for twistors [62], particles and superparticles, instanton and gauge field, etc. Bandos, et al., even proposed supertwistors [63]. The variables of supertwistors are:

$$Z_L^I = (\lambda_L^\alpha, \mu_L^{\dot{\alpha}}, \psi_L^A), \quad Z_R^I = (\lambda_R^\alpha, \mu_R^{\dot{\alpha}}, \psi_R^A). \quad (14)$$

These are a pair equations of three variables. Generally, these are n pair equations of m variables. Fedoruk, et al., researched unification of various string models from twistor [64], and twistor string, etc [65].

We researched applications of twistor and its extensions in biology, which may describe some biological duality, and proposed specially the twistor model of DNA [66]. It is well-known that space-time depend on matter and its movement in general relativity. Interactions between biological elements of different levels determine biological structures and shapes. For example, in DNA "horizontal" hydrogen bond interaction connects A-T, and "vertical" stacking interaction connects C-G [67]. Bena, et al., [68,69] elucidated the one-loop twistor-space structure corresponding to momentum-space maximally helicity-violating diagrams, and use the "holomorphic anomaly" to define modified differential operators which can be used to probe the twistor-space structure of one-loop amplitudes [68,69]. Bena, et al., searched loops in twistor space, and twistor transform in d dimensions and a unifying role for twistors [69]. We applied the loop quantum theory to biology, and proposed the model of protein folding and lungs, and obtain four approximate conclusions [70]. Further, it may combine twistor [66].

So far, biology applies only twister. Twistor as an extension of scalar, vector, spinor and tensor includes the self-dual Yang-Mills field. We may research generally twistor in biology. Based on complex number and the conformal transformation, twistor is [54]:

$$\begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} = t\sigma_0 + x\sigma_x + y\sigma_y + z\sigma_z. \quad (15)$$

Here using quaternion or Pauli matrices are:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (16)$$

Twistor (11) is two pair conjugate complex numbers, and includes (t, x, y, z) and Pauli matrices. Its space is C^4 . This may be related to relativity. Pauli matrices describe particles with spin 1/2.

Some structures in biology are helix. RNA is a single link structure on A-U and G-C. Here A is adenine and G is guanine, while U is uracil and C is cytosine. It is known that a helical line is:

$$x = a \cos t, y = b \sin t, z = dt. \quad (17)$$

Here d is a thread pitch. A form of the complex function is:

$$z = a \cos t + ib \sin t, w = dt. \quad (18)$$

The double helix is:

$$z = a \cos(\varphi + \frac{C_1}{C_2}) \text{ and } w = d\varphi. \quad (19)$$

In complex function $f^{x+iy} = f^x f^{iy}$. When $f=e$, so $e^x e^{iy}$ is a circle with radius $\rho = e^x$ and $\phi = y$ [71].

Twistor may describe biological extensive string with meander and twister. RNA as a string, and DNA as the double string of the double helix structure on A-T and G-C (here T is thymine) should be able to apply twistor. Therefore, we may research the twistor model described DNA, RNA, etc [66].

We may transform quaternion or Pauli matrices to other different forms:

$$1) 1 \rightarrow I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ is unit matrix, and } i \rightarrow C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } C^2 = -I. \text{ Assume}$$

that a new complex positive number j, here $j^2=1$. Then, let $j \rightarrow B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$B^2 = I$, correspond to an inversion operator. $k = ij \rightarrow A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = CB$, here

$A^2 = I$ and $ji = -k \rightarrow -A$. This set (1,i,j,k) is another quaternion, and relates the Clifford fourfold quaternion. It corresponds to that a field is extended to a ring. Such new twistor is:

$$\begin{pmatrix} t+z & x+y \\ x-y & t-z \end{pmatrix} = tI + xB - yC - zA. \quad (20)$$

2) We introduce two pair conjugate complex numbers

$$\begin{pmatrix} t + iz & x + iy \\ x - iy & t - iz \end{pmatrix} = tI + xB - iyC - izA. \quad (21)$$

This corresponds to two circles.

3) We may add four parameters

$$\begin{pmatrix} at + dz & bx + cy \\ bx - cy & at - dz \end{pmatrix} = atI + bx B - cy C - dz A. \quad (22)$$

Such their radii may be different, and may be determined. For DNA they correspond to a diameter 20A and the pitch 34A, etc.

These corresponding relations between new quaternion and DNA ($t \sim A$, $z \sim T$ and $x \sim G$, $y \sim C$) are symmetry completely. It is usual B-DNA. Other is A-DNA and Z-DNA, etc. New forms of twistor correspond to develop matrix and tensor, etc.

Twistor is developed from general relativity, and may describe the curved space-time. It is also consistent with a basic idea: the matter determines the structure of space in biology [66]. Moreover, A-T(U), C-G in DNA (RNA) all are 2 square matrix.

Moreover, in biology there is duality. For example, duality and synergy in peptide antibiotic mechanisms by which peptide antibiotics disrupt bacterial DNA synthesis, protein biosynthesis, cell wall biosynthesis, and membrane integrity shown rich diversity, and involved in synergistic relations with antibiotics and proteins [72]. Two ideas in theoretical biology, 'decomposition into functions' and 'gluing functions', show a duality. They imply two biologically significant conditions: the existence of cycles in finite graphs and anticipatory diagrams [73]. Al-Sady, et al., searched mechanistic duality of transcription factor function in phytochrome signaling, and found that PIF3 acts positively as a transcription factor, exclusively requiring its DNA-binding capacity [74]. Twistor and spinor with double components may describe some biological duality, for example, excitatory synapse and inhibitory synapse, and above duality. The twistor model of DNA [66] is probably advantageous to development of DNA computing model [75].

4. Some Applications of Various Fields

New researches shown that the entangled state should be is a new fifth interaction [76]. Its action distance seem to be middle-rang, and its strength is also middle one. This likes the thought field [77]. Further, we proposed the entangled field, whose phase particle (phason) has some characters and corresponding equations. It is tachyon, and assume that it is similar to photon and $J=1$ and $m=0$ or mass is very small as similar neutrino, and may show the action at a distance. We researched that this field as wave has some characters, and discussed the superluminal quantum communication by a pair

of entangled states is generated on both positions, or by preparing and transmitting a pair of entangled instruments, so the superluminal quantum communication. Assume that the entangled field has a similar magnetic theory, which may be a quantum cosmic field, or be the extensive quantum theory, or God or the Buddha-fields and so on. These are all macroscopic fields, which correspond to de Broglie-Bohm nonlinear "hidden variable" theory, but it is microscopic [78].

There are Earth's magnetic field [79] and stress field of the Earth's crust [80]. These fields are related to the plate tectonics and global mantle currents. They also determine earthquakes. Based on the nonlinear equations of fluid dynamics, we derived a simplified nonlinear solution of momentum, which correspond to the accumulation of the energy. Their values are excess a faulting threshold, earthquake will take place. From this a chaos equation is obtained, in which chaos corresponds to the earthquake, which shows complexity on seismology, and impossibility of exact prediction of earthquakes. But, combining the Carlson-Langer model and the Gutenberg-Richter relation, we derived approximately the magnitude-period formula of the earthquakes [17,77,81-84]:

$$T = 10^{-b(M_0-M)} T_0. \quad (23)$$

By this formula some predictions can be calculated quantitatively and are already tested. We forecasted a series of earthquakes in California [77,82,83] since 2004. From this in California next earthquakes should take place in 2009, 2014 and 2019, etc. Earthquakes (M=6.9) occurred on 3 August of 2009, and M=6 on 24 August of 2014. A large earthquake of 2019 is more possible [77,82,83]. In fact, two earthquakes occurred July 4 (M=6.4) and July 5 (M=7.1) in California, which agrees completely with my prediction [84].

Mathematics is a powerful and important tool in modern ecology and environment science [85,86]. We researched field theory is applied to social sciences [2]. Ecological field is namely useful concept. Its complete mode is the unification field of human-nature in Chinese traditional culture. Using the similar formulas of the preference relation and the utility function, we proposed the confidence relations and the corresponding influence functions that represent various interacting strengths of different families, cliques and systems of organization. It produces a multiply connected topological economics. This model may describe a corruption field in usual economic system. Further, we discussed the binary periods of the political economy by the complex function and the elliptic functions [87].

We proposed the social extensive general relativity. Everyone possesses fate and luck, and both correspond to mass self and life orbit and movement. General relativity shows that matter and movement determine the space-time of everyone, and may derive the corresponding field equations and the evolutional orbits. In Fig. 2 the big mass of the center and its movement

correspond to the great countries and great men that determine space-time and era, from which everyone's mass and efforts determine the orbits of life. Both determine the evolution of whole society and mankind. This as a universal physical representation of causality is a great contribution to modern social science. It is the causality field as a common basis of various natural sciences, Buddhism and some social sciences [88].



Fig. 2. Matter and movement determine the space-time, and space-time determines the evolutionary orbits.

Based on the social structure we introduced the social individual-wave duality, and researched the social topology and the social strain field [86]. A variant of the damage field $D(r,t)$ should agree with the damage field equation:

$$\frac{\partial D}{\partial t} + \frac{\partial(Dv)}{\partial r} = f. \quad (24)$$

Here f is dynamical function of damage.

In a word, in natural science and social science there are widely change and evolutionary fields. The development of mathematics often leads to the progress of physics and science. Field theory has been widely applied in many regions of natural and social sciences, and any development of field theory will necessarily inspire and apply to more aspects.

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**A CHARACTERIZATION OF NON TWISTED AFFINE LIE
ALGEBRAS FROM GENERALIZED STRUCTURABLE ALGEBRAS**

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Abstract

Generalized structurable algebras contain the class of Lie algebras, alternative algebras, Poisson algebras, and a class of nonassociative algebras appeared in mathematical physics and they are therefore significant for applications. In this paper we investigate the construction of affine Lie algebras from generalized structurable algebras.

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§ 0. Introduction

In this article, we shall study a construction of non-twisted affine Lie algebras as generalized structurable algebras explicit. This implies that an explicit formula of the affine Lie algebra is given by means of standard embedding Lie algebras and its construction is investigated.

In the recent paper [2] we generalized the notion of a structurable algebra. That is, in the cited paper, we investigated a generalization of a class of well known nonassociative algebras containing Lie algebras, Jordan algebras, structurable algebras, Lie isotopic algebras, etc. This generalized class, called a generalized structurable algebra, is important from the point of view of Lie algebras construction, which has been studied in the construction of finite dimensional simple Lie algebras. As an example of this phenomenon and first purpose, we shall construct a generalized structurable algebra from a pair of structurable algebras (Section 2).

The secondary purpose of this article is to give a unified treatment for affine Lie algebras by means of standard embedding Lie algebras and to show how the standard embedding can be constructed from our generalized structurable algebras. More explicitly speaking, we shall investigate a construction of Virasoro algebras and affine Lie algebras from generalized structurable algebras (Section 3 and Section 4).

In final section, we shall describe to "isotopy" for generalized structurable algebras (Section 5).

Throughout this paper, we shall be concerned with algebras over a field Φ not of characteristic 2 or 3 and do not assume that our algebras are finite dimensional, unless otherwise specified.

§ 1. Definition and Preliminaries

In this section, we shall exhibit several examples of generalized structurable algebras. First we define a *generalized structurable algebra* over a field Φ of $ch \Phi \neq 2$ or 3 to be a nonassociative algebra A equipped with a bilinear derivation $D(x, y) (\neq 0)$ in which the following conditions are satisfied

$$D(x, y) = -D(y, x) \quad (1-1)$$

$$D(xy, z) + D(yz, x) + D(zx, y) = 0 \quad (1-2)$$

for all $x, y, z \in A$.

Example 1. Let $(A, [,])$ be a Lie algebra over Φ . Then A is a generalized structurable algebra equipped with a derivation

$$D(x, y) := ad[x, y].$$

Example 2. Let $(A, \circ)$ be a commutative Jordan algebra over Φ . Then A is a generalized structurable algebra equipped with a derivation

$$D(x, y) := [L(x), L(y)]$$

where

$$[L(x), L(y)]z = x \circ (y \circ z) - y \circ (x \circ z).$$

Example 3. Suppose $(A, \bar{})$ is a unital algebra with an involution $\bar{}$ over Φ . $(A, \bar{})$ is said to be *structurable* if

$$[L(x, y), L(z, w)] = L(\langle xyz \rangle, w) - L(z, \langle yxw \rangle)$$

where

$$L(x, y)z = \langle xyz \rangle := (x\bar{y})z + (z\bar{y})x - (z\bar{x})y$$

for x, y, z in A . (cf. [1])

Then A is a generalized structurable algebra equipped with a derivation

$$D(x, y) := \frac{1}{3}([x, y] + [\bar{x}, \bar{y}], z) + [z, y, x] - [z, \bar{x}, \bar{y}] \quad (1-3)$$

where

$$[x, y] := xy - yx, [x, y, z] := (xy)z - x(yz).$$

In fact, the identities (1-1) and (1-2) are valid for the structurable algebra A . Thus any structurable algebra is a generalized structurable algebra. This derivation $D(x, y)$ induced by (1-4) is said to be an *inner derivation* of the structurable algebra. The set of all such derivation is denoted by $\text{Inn Der } A$.

From now on, in a generalized structurable algebra A equipped with a derivation $D(x, y)$, we also denote by $\text{Inn Der } A$, the subspace spanned by all the $D(x, y)$.

Theorem 1.1([2][3]) *Let A be a generalized structurable algebra equipped with a derivation $D(x, y)$ satisfying $\sum_{(x,y,z)} (D(x, y)z + [[x, y], z]) = 0$ and $[D(z, w), D(x, y)] = D(D(z, w)x, y) + D(x, D(z, w)y)$ for all $x, y, z, w \in A$. Then the vector space $L(A) := \text{Inn Der } A \oplus A$ is a Lie algebra with respect to the new bracket operation $[\ , \]^*$*

$$[D+x, E+y]^* := [D, E] + D(x, y) + Dy - Ex + [x, y] \quad (1-4)$$

for $D, E \in \text{InnDer } A, x, y \in A$.

The Lie algebra obtained from this Theorem is said to be the *standard embedding Lie algebra* associated with a generalized structurable algebra.

Remark ([2]) If we consider the following cases of Lie algebra's construction;

(i) the first Tit's construction;

(ii) the second Tit's construction;

then we note that the above two constructions of Lie algebras may be characterized by means of our concept (Theorem 1.1) via the generalized structurable algebras.

Remark ([3]) Let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be a decomposition into the subspace L_i of a Lie algebra L satisfying

$$[L_i, L_j] \subset L_{i+j} \quad (\text{if } |i+j| \leq 2)$$

$$[L_i, L_j] = 0 \quad (\text{if } |i+j| \geq 3)$$

$$\sum_{i \neq 0} [L_i, L_{-i}] = L_0$$

for all integer $-2 \leq i, j \leq 2$.

Then $A = L_{-2} \oplus L_{-1} \oplus L_1 \oplus L_2$ is a generalized structurable algebra with respect to the product

$$X \circ Y := [x_{-2}, y_1] + [x_{-1}, y_{-1}] + [x_{-1}, y_2] +$$

$$[x_1, y_{-2}] + [x_1, y_1] + [x_2, y_{-1}]$$

and the derivation

$$D(X, Y) := ad([x_{-2}, y_2] + [x_{-1}, y_1] + [x_1, y_{-1}] + [x_2, y_{-2}])$$

$$X = x_{-2} + x_{-1} + x_1 + x_2, \quad Y = y_{-2} + y_{-1} + y_1 + y_2$$

$$\text{for } x_i, y_i \in L_i.$$

As a generalization of Theorem 1.1, we have following theorem.

Theorem 1.2.(Extended property for generalized structurable algebra) *Let A be a generalized structurable algebra equipped with a derivation $D(x, y)$ such that the derivations satisfy the relation $[D, D(x, y)] = D(Dx, y) + D(x, Dy)$, for all $D, D(x, y) \in \text{InnDer } A$. Then the vector space $B := \text{InnDer } A \oplus A$ is a generalized structurable algebra equipped with new product and new derivation defined by*

$$[X, Y]^* := D_1 x_2 - D_2 x_1 + [x_1, x_2] + [D_1, D_2] + D(x_1, x_2)$$

$$D^*(X, Y)Z :=$$

$$[[D_1, D_2], D_3] + [D(x_1, x_2), D_3] + D(x_1, x_2)x_3 + [D_1, D_2]x_3$$

for $X = D_1 + x_1, Y = D_2 + x_2, Z = D_3 + x_3, D_i \in \text{InnDer } A, x_i \in A$.

Proof. From the definitions of the product $[,]^*$ and the derivation $D(X, Y)$, it is enough to show that the identities

$$D^*(X, Y) = -D^*(Y, X),$$

$$\sum_{(X, Y, Z)} D^*([X, Y]^*, Z)W = 0,$$

$$D^*(X, Y)[Z, W]^* = [D^*(X, Y)Z, W] + [Z, D^*(X, Y)W].$$

As these identities are verified by straightforward calculations by long, we omit it. This completes the proof.
Remark We note that results in this section can be generalized to super or graded concepts (for example, to see [3],[4],[5]).

§ 2. Construction of finite dimensional simple Lie algebras

In this section, we shall give finite dimensional simple Lie algebras as an example of the standard embedding of generalized structurable algebras.

Let A be a structurable algebra (cf. Example 3 of Section 1).

We put

$$S := \{s \in A \mid \bar{s} = -s\},$$

$$N := \{(x, s) \mid x \in A, s \in S\},$$

$\tilde{N} :=$ a copy of N , and $B := N \oplus \tilde{N}$ (as vector space).

Then we have the following

Theorem 2.1 *Let B be as in above. Then B is a generalized structurable algebra equipped with a product*

$$XY := (r_1 y_2 - r_2 y_1, x_1 \bar{x}_2 - x_2 \bar{x}_1) + (s_1 x_2 - s_2 x_1, y_1 \bar{y}_2 - y_2 \bar{y}_1) \sim (2-1)$$

and a derivation

$$D(X, Y) := L(x_1, y_2) - L(x_2, y_1) + L(r_1)L(s_2) - L(r_2)L(s_1), (2-2)$$

under the action of $L(x, y)$ and $L(s)L(t)$ on $B = N \oplus \tilde{N}$ as follows,

$$\begin{aligned} L(x, y)(z, r) &= (L(x, y)z, (ry)\bar{x} + x(\bar{y}r)) \\ L(x, y)(z, r)^\sim &= (-L(y, x)z, -(rx)\bar{y} - y(\bar{x}r)) \\ L(s)L(t)(z, r) &= (s(tz), s(tr) + r(ts)) \\ L(s)L(t)(z, r)^\sim &= (-t(sz), -t(sr) - r(st)) \end{aligned}$$

where

$$\begin{aligned} X &= (x_1, r_1) + (y_1, s_1)^\sim, Y = (x_2, r_2) + (y_2, s_2)^\sim \in N \oplus \tilde{N}, \\ L(x, y)z &= \langle xyz \rangle, L(r)x = rx. \end{aligned}$$

Proof. From the definition of the product B and the derivation $D(X, Y)$, we can verify this proof by straightforward but very long calculations and we omit it.

In this generalized structurable algebra B , we have the following identities

$$\begin{aligned} &[D(X, Y), D(Z, W)] \\ &= D(D(X, Y)Z, W) + D(Z, D(X, Y)W) \\ &\sum_{(X, Y, Z)} (D(X, Y)Z + (XY)Z) = 0 \end{aligned}$$

for all $X, Y, Z, W \in B$. Thus we can obtain the standard embedding Lie algebra associated with B satisfying the following Lie products;

$$[(x, s), (y, t)] = (0, x\bar{y} - y\bar{x})$$

$$\begin{aligned}
 [(x, s)^\sim, (y, t)^\sim] &= (0, x\bar{y} - y\bar{x})^\sim \\
 [(x, s), (y, t)^\sim] &= (-tx, 0)^\sim + L(x, y) + L(s)L(t) + (sy, 0) \\
 [T_z + d, (x, s)] &= (T_z x + dx, zs + s\bar{z} + ds) \\
 [T_z + d, (x, s)^\sim] &= (-T_{\bar{z}} x + dx, -\bar{z}s - sz + ds)^\sim
 \end{aligned}$$

where

$$T_z := L(z, 1), z \in A \text{ and } d \in \text{Inn Der } A.$$

For this Lie algebra $L(B) = \text{InnDer } B \oplus B$, we have the following decomposition;

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \quad (2-3)$$

satisfying

$$[L_i, L_j] \subset L_{i+j} (\text{if } |i+j| \leq 2), [L_i, L_j] = 0 (\text{if } |i+j| \geq 3)$$

where

$$L_{-1} = A, L_{-2} = S, L_1 = A^\sim, L_2 = S^\sim, \quad (2-4)$$

$$\begin{aligned}
 L_0 &= \text{InnDer } B = \{D(X, Y) | X, Y \in B\}_{\text{span}} \\
 &= \{L(x, y) | x, y \in A\}_{\text{span}} \\
 &= T_A \oplus \text{Inn Der } A := \text{Inn Strl}(A, ^\sim). \quad (2-5)
 \end{aligned}$$

Remark. By the choice of central simple structurable algebras (for example, to see [1]), we can obtain isotopic

central simple Lie algebras via the construction of the over standard embedding.

On the other hand, by a straightforward calculation we may give a characterization of a Lie triple system via a generalized structurable algebra as appeared in following Lemma.

Lemma 2.2. *Let $(T, [-, -, -])$ be a Lie triple system. Then T is a generalized structurable algebra equipped with a product*

$$XY = 0 \text{ (identically zero)}$$

and a derivation

$$D(X, Y)Z = [XYZ].$$

This derivation $D(X, Y)$ is an inner derivation of T , that is, satisfies the identity

$$[XY[ZUV]] = [[XYZ]UV] + [Z[XYU]V] + [ZU[XYV]].$$

Thereby, as a variation of Theorem 1.1, we have the following.

Proposition 2.3. *Let T be as in Lemma 2.2. Then the vector space*

$$L(T) := \text{Inn Der } T \oplus T$$

is a Lie algebra with respect to the bracket operation

$$[D + x, E + y]^* := [D, E] + D(x, y) + Dy - Ex.$$

This is called to the standard embedding Lie algebra associated with a Lie triple system (for example, to see [6]).

§ 3. Construction of Virasoro algebras

In this section, we shall characterize a Virasoro algebra as a generalized structurable algebra.

By the same methods as in section 1, we have the following:

Theorem 3.1 *Let $A := \bigoplus_{j \in \mathbb{Z}} C d_j$ be a Lie algebra over the complex number field C equipped with the product*

$$[d_i, d_j] = (j - i)d_{i+j}.$$

Then A is a generalized structurable algebra with a derivation

$$D(d_i, d_j) := \frac{1}{12}(j^3 - j)\delta_{i,-j}c,$$

where c is an element such that $c = ad\ b$ and b satisfies $[b, x] = 0$ for all $x \in A$.

Furthermore, $L(A) = \text{Inn Der } A \oplus A$ is a Lie algebra with respect to the new product

$$[D + x, E + y]^* := [D, E] + D(x, y) + Dy - Ex + [x, y]$$

for

$$D, E \in \text{Inn Der } A = C\{c\}, \quad x, y \in A.$$

Proof. Since other identities in the above vector space A and $L(A)$ are trivial, it is enough to show that the identities;

$$\sum_{(i,j,k)} D([d_i, d_j], d_k) = 0 \quad (3-1)$$

$$\sum_{(i,j,k)} [[d_i + \mu_i c, d_j + \mu_j c]^*, d_k + \mu_k c]^* = 0 \quad (3-2)$$

The proof of (3-1):

$$\begin{aligned} \sum_{(i,j,k)} D([d_i, d_j], d_k) &= \sum_{(i,j,k)} (j-i) D(d_{i+j}, d_k) \\ &= \frac{1}{12} \sum_{(i,j,k)} (j-i)(k^3 - k) \delta_{i+j, -k} c = 0 \end{aligned}$$

The proof of (3-2):

From the identity

$$[d_i + \mu_i c, d_j + \mu_j c]^* = (j-i) d_{i+j} + \frac{1}{12} (j^3 - j) \delta_{i, -j} c,$$

we have

$$\begin{aligned} &[[d_i + \mu_i c, d_j + \mu_j c]^*, d_k + \mu_k c]^* \\ &= (j-i)(k - (i+j)) d_{i+j+k} \\ &\quad + \frac{(j-i)}{12} (k^3 - k) \delta_{i+j, -k} c. \end{aligned}$$

Thus we obtain

$$\begin{aligned} &\sum_{(i,j,k)} [[d_i + \mu_i c, d_j + \mu_j c]^*, d_k + \mu_k c]^* \quad (3-3) \\ &= \{(j-i)(k - (i+j)) + (k-j)(i - (j+k)) + \end{aligned}$$

$$(i-k)(j-(k+i))\}d_{i+j+k} + \frac{1}{12}\{(j-i)(k^3-k)\delta_{i+j,-k} + \\ (k-j)(i^3-i)\delta_{j+k,-i} + (i-k)(j^3-j)\delta_{k+i,-j}\}c$$

(i) The case of $i+j+k=0$. The identity(3-3) is the following.

The relation(3-3)=

$$\frac{1}{12}\{(j-i)(k^3-k) + (k-j)(i^3-i) + (i-k)(j^3-j)\} = 0$$

(ii) The case of $i+j+k \neq 0$. The identity(3-3) is the following.

The relation(3-3)=

$$(j-i)(k-(i+j)) + (k-j)(i-(j+k)) + (i-k)(j-(k+i)) = 0..$$

This completes the proof.

Remark. This theorem 3.1 means that the derivation of A (in the sense of generalized structurable algebra) coincides with the center of A (in the sense of Lie algebra), and that $L(A)$ may be constructed by applying of the concept of Theorem 1.1., that is, by means of the standard embedding.

Remark. We note that this Lie algebra $L(A) = \text{Inn Der } A \oplus A$ is referred to as the Virasoro algebra, which plays an important role in mathematical physics.

§ 4. Construction of affine Lie algebras

In this section, by making use of the standard embedding, we shall characterize non-twisted affine Lie algebras from the generalized structurable algebras.

Let g be a finite dimensional Lie algebra and $L = C[t, t^{-1}]$ be the algebra of Laurent polynomials in t over the complex number field C .

On $L \otimes g$, we may define a structure of a generalized structurable algebra as follows.

Proposition 4.1 *Let g be a Lie algebra and $L = C[t, t^{-1}]$ be the algebra of Laurent polynomials in t over the complex number field C . Then $L \otimes g$ is a generalized structurable algebra equipped with a product and a derivation;*

$$[t^k \otimes x, t^{k_1} \otimes y] = t^{k+k_1} \otimes [x, y] \quad (4-1)$$

$$d(t^k \otimes x, t^{k_1} \otimes y)t^{k_2} \otimes z = \frac{1}{2}(k - k_1)\delta_{k, -k_1}k_2t^{k_2} \otimes (x, y)z. \quad (4-2)$$

where (x, y) is the Killing form of g .

Proof. By straightforward calculations, we get the following identities;

$$d(X, Y) = -d(Y, X)$$

$$\sum_{(X, Y, Z)} d([X, Y], Z)W = 0$$

$$d(X, Y)[Z, W] = [d(X, Y)Z, W] + [Z, d(X, Y)W],$$

$$\text{for } X, Y, Z, W \in L \otimes g.$$

Thus these imply that $L \otimes g$ is a generalized structurable algebra. This completes the proof.

We denote by $\text{Inn Der}(L \otimes g)$, the subspace spanned by all derivations defined by identity (4-2).

Therefore by making use of the concept of the standard embedding Lie algebra, we have the following.

Proposition 4.2. *Let $L \otimes g$ be as in the above Proposition 4.1. Then the vector space $\text{InnDer}(L \otimes g) \oplus L \otimes g$, where $\text{InnDer}(L \otimes g) := \{d(X, Y) | X, Y \in L \otimes g\}_{\text{span}}$ is the standard embedding Lie algebra associated with $L \otimes g$. That is, with respect to the bracket*

$$[d' + X, d'' + Y]^* := [d', d''] + d'Y - d''X + [X, Y]$$

$$\text{for } d', d'' \in \text{InnDer}(L \otimes g), X, Y \in L \otimes g.$$

Remark. If $k = -k_1 = 1$ and $(x, y) = 1$, then we have

$$d(t \otimes x, t^{-1} \otimes y)t^{k_2} \otimes z = k_2 t^{k_2} \otimes z.$$

Thus this derivation algebra $\text{InnDer}(L \otimes g)$ contains an element d such that $d : t^k \otimes x \longrightarrow kt^k \otimes x$. Conversely, from the definition of $\text{InnDer}(L \otimes g)$ it is easily shown that Cd contains $\text{InnDer}(L \otimes g)$. Therefore we get $Cd = \text{InnDer}(L \otimes g)$.

We put $A := L \otimes g \oplus Cd$, where d is the derivation of $L \otimes g$ such that $d : t^k \otimes x \longrightarrow kt^k \otimes x$. This algebra A coincides with the Lie algebra $\text{InnDer}(L \otimes g) \oplus L \otimes g$.

More explicitly, this vector space A is a Lie algebra with respect to the product

$$\begin{aligned} & [t^k \otimes x \oplus \mu d, t^{k_1} \otimes y \oplus \mu_1 d]^* : \\ & = t^{k+k_1} \otimes [x, y] + \mu k_1 t^{k_1} \otimes y - \mu_1 k t^k \otimes x. \end{aligned}$$

Thus we have the following

Theorem 4.3 *Let A be as in above. Then A is a generalized structurable algebra equipped with a derivation*

$$D(t^k \otimes x \oplus \mu d, t^{k_1} \otimes y \oplus \mu_1 d) := k\delta_{k, -k_1}(x, y)c,$$

where (x, y) is the Killing form of g and c is an element such that $c = ad\ b$ and b satisfies $[b, X]^* = 0$ for all $X \in A$.

Proof. It remains to show that

$$D(X, Y) = -D(Y, X) \quad (4-3)$$

$$\begin{aligned} D(X, Y)[Z, W]^* = \\ [D(X, Y)Z, W]^* + [Z, D(X, Y)W]^* \end{aligned} \quad (4-4)$$

$$\sum_{(X, Y, Z)} D([X, Y]^*, Z) = 0, \quad (4-5)$$

where $X, Y, Z, W \in A = L \otimes g \oplus Cd$.

From the definition of the derivation, it is clear that identities (4-3) and (4-4) hold.

The proof of (4-5):

From the identity

$$D([X, Y]^*, Z) =$$

$$D(t^{k+k_1} \otimes [x, y] + \mu k_1 t^{k_1} \otimes y - \mu_1 k t^k \otimes x, t^{k_2} \otimes z \oplus \mu_2 d),$$

where

$$X = t^k \otimes x \oplus \mu d, Y = t^{k_1} \otimes y \oplus \mu_1 d, Z = t^{k_2} \otimes z \oplus \mu_2 d,$$

we obtain

$$D([X, Y]^*, Z) = \{(k + k_1)\delta_{k+k_1, -k_2}([x, y], z) + \mu k_1^2 \delta_{k_1, -k_2}(y, z) - \mu_1 k^2 \delta_{k, -k_2}(x, z)\}c.$$

Therefore from the property of the Killing form, it follows that

$$\sum_{(X, Y, Z)} D([X, Y]^*, Z) = 0.$$

This completes the proof.

Remark This theorem 4.3 means that the space of derivations of A (in the sense of generalized structurable algebra) coincides with the center of $L(A)$ (in the sense of Lie algebra).

From this theorem 4.3, we have the following by applying of the concept of Theorem 1.1.

Theorem 4.4 *Let A be the generalized structurable algebra as in theorem 4.3. Then $L(A) := \text{InnDer} A \oplus A$ is a Lie algebra with respect to the new product*

$$\begin{aligned} & [t^k \otimes x \oplus \mu d \oplus \lambda c, t^{k_1} \otimes y \oplus \mu_1 d \oplus \lambda_1 c]^{**} : \\ & = (t^{k+k_1} \otimes [x, y] + \mu k_1 k^{k_1} \otimes y - \mu_1 k t^k \otimes x) \oplus k \delta_{k, -k_1}(x, y)c \\ & \text{for } \lambda c, \lambda_1 c \in \text{Inn Der } A = C\{c\}, t^k \otimes x \oplus \mu d, t^{k_1} \otimes y \oplus \mu_1 d \in A. \end{aligned}$$

Proof. From the definition of the generalized structurable algebra, it follows that the identities

$$D(X, Y)Z = 0, D(Y, Z)X = 0, D(Z, X)Y = 0,$$

$$\sum_{(X, Y, Z)} [[X, Y]^*, Z]^* = 0$$

are verified for $X, Y, Z \in A$. Thus $L(A)$ is a Lie algebra.

Remark. These results imply that the affine Lie algebras can be constructed by means of the standard embedding Lie algebras associated with the generalized structurable algebras via the concept of Theorem 1.1., that is, by a unified method. (:= by means of standard embedding)

§ 5. Isotopy of generalized structurable algebras

In this final section, we shall note a few word to the concept of the "isotopy" for our generalaized structurable algebras.

- Let A be an assosiative algebra over a field Φ . Then A becomes a Lie algebra with respect to new product

$$[x, y] := xy - yx.$$

- Let A be an alternative algebra over a field Φ . Then A becomes a Malcev algebra with respect to new product

$$[x, y] := xy - yx.$$

For the "isotopy" of a generalized structurable algebra we have the following.

Theorem 5.1. (closed property of "isotopy" for generalized structurable algebra) *Let A be a generalized structurable algebra equipped with a derivation $D(x, y)$ over a field Φ . Then A becomes a generalized structurable algebra with respect to new product $[,]_q$ and the derivation $D(x, y)^*$*

$$[x, y]_q := xy - qyx$$

$$D^*(x, y) := D(x, y),$$

where q is a scalar element such that $qD^*(x, y) = D^*(qx, y) = D^*(x, qy)$.

For class of nonassociative algebras, we have,

- Lie algebra $\subset$ Malcev algebra $\subset$ generalized structurable algebra,
- associative algebra $\subset$ alternative algebra $\subset$ generalized structurable algebra,
- commutative algebra $\subset$ accessible algebra $\subset$ generalized structurable algebra.

These imply that our algebras are wide class.

Remark. In my judgement, the proposers of "isotopy" are Prof.A.Albert (in Mathematics), and Prof.R.M.Santilli (in Physics), furthermore, it seems that the concept of "isotopy" is very valuable in study for mathematical physics, quantumn mechanics, theoretical physics and natural sciences.

§ 6. Concluding Remarks.

In final remark, briefly we shall summarize the results of this article as follows:

generalized structurable algebras

↓
as standard embedding

reductive Lie algebras (finite or infinite dimensional)

↓ as wide class

generalized structurable algebras

↓ as isotopy

generalized structurable algebras.

On the others hand, if we are admitted to talk roughly,
we may describe to our concept as follows;

V : vector space , manifold and tangent space

↓

$D(V, V)$: Lie algebras of the derivations of V

↓

$L(V) := D(V, V) \oplus V$: the standard embedding of V

↓

V : generalized structurable algebra and

$L(V)$: Lie algebra

↓

V : algebra or fields on Physics,

$D(V, V)$: action or operator on V

These are reasons for studying our generalized structurable algebras.

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